Chapter 5

Chapter 5: Introduction to Digital Filters

5.1 Introduction

There are two types of digital filters:

1. Recursive (there is at least one feedback path in the filter)
2. Non-recursive (no feedback paths)

A linear time invariant (LTI) discrete system described by the following equation is commonly called a digital filter:

$$y[n] = \underbrace{\sum_{k=0}^{M} a_k x[n-k]}_{\text{Feed forward}} - \underbrace{\sum_{k=1}^{L} b_k y[n-k]}_{\text{Feedback}} \qquad (5.1)$$

where $x[n]$ is the input signal, $y[n]$ is the output signal. $a_0, a_1, a_2, \ldots, a_M; b_1, b_2, b_3, \ldots, b_L$ are constants (filter coefficients). These coefficients determine the characteristics of the system.

when $b_k = 0$	the filter is said to be non-recursive type
when $b_k \neq 0$	recursive type.

5.1.1 Non recursive digital filters (FIR)

If $b_k = 0$, then the calculation of $y[n]$ does not require the use of previously calculated samples of the output (see equation (5.1)).

$$y[n] = \sum_{k=0}^{M} a_k x[n-k] = a_0 x[n] + a_1 x[n-1] + \cdots + a_M x[n-k]$$

This is recognised as a convolution sum.

$$y[n] = \sum_{k=0}^{M} h[k] x[n-k]$$

Therefore the impulse response is identical to the coefficients, that is,

$$h[n] = \begin{cases} a_n & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

$$y[n] = \sum_{k=0}^{M} a_k x[n-k] = \underset{\underset{h_0}{\uparrow}}{a_0} x[n] + \underset{\underset{h_1}{\uparrow}}{a_1} x[n-1] + \cdots + \underset{\underset{h_M}{\uparrow}}{a_M} x[n-k]$$

Any filter that has an impulse response of finite duration is called Finite Impulse Response (FIR) filter.

<u>Example:</u>

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2]$$

$$H(z) = \frac{Y(z)}{X(z)} = a_0 + a_1 z^{-1} + a_2 z^{-2}$$

This is a (non-recursive) second order FIR filter

A property of the FIR filter is that it will always be stable.

(a) Stability requires that there should be no poles outside the unit circle. This condition is automatically satisfied since there are no poles at all outside the origin. (In fact, all poles are located at the origin.)

(b) Another property of non-recursive filter is that we *can* make filters with exactly linear phase characteristics

Note: The ability to have an exactly linear phase response is one of the most important properties of a LTD system (filter). When a signal passes through a filter, it is modified in amplitude and/or phase. The nature and extent of the modification of the signal is dependent on the amplitude and phase characteristics of the filter.

The **phase delay** or **group delay** of the filter provides a useful measure of how the filter modified the phase characteristic of the signal. If we consider a signal that consists of several frequency components (eg. speech waveform) the phase delay of the filter is the amount of time delay each frequency component of the signal suffers in going through the filter.

$$\boxed{phase\_delay(T_p) = -\frac{\phi(\theta)}{\theta} \qquad (5.2)}$$

Note: T_p = the negative of the phase angle divided by frequency

The group delay on the other hand is the average time delay the composite signal suffers at each frequency as it passes from the input to the output of the filter.

$$\boxed{group\_delay(T_g) = -\frac{d\phi(\theta)}{d\theta} \qquad (5.3)}$$

Note: T_g = the negative of the derivative of the phase with respect to frequency]

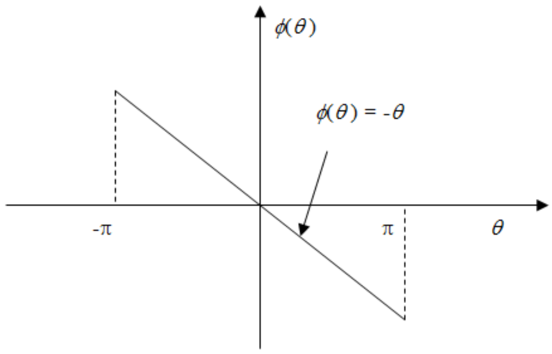
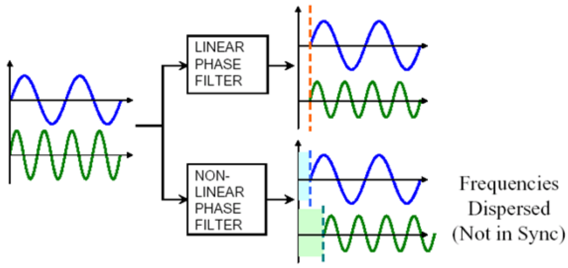


Figure 5.1: Phase response of a linear phase filter



A diagram comparing the performance of a linear phase filter and a non-linear phase filter

A *constant group delay* means that signal components at different frequencies receive the same delay in the filter.

A *linear phase filter* gives same time delay to all frequency components of the input signal. A filter with a nonlinear phase characteristic will cause a phase distortion in the signal that passes through it.

This is because the frequency components in the signal will each be delayed by an amount not proportional to frequency, thereby altering their harmonic relationship. Such a distortion is undesirable in many applications, for example music, video etc.

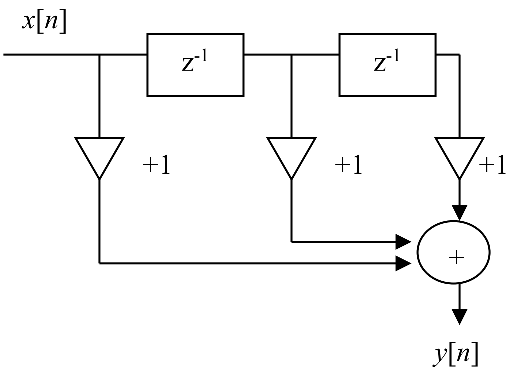
A filter is said to have a linear phase response if its phase response satisfies one of the following relationships:

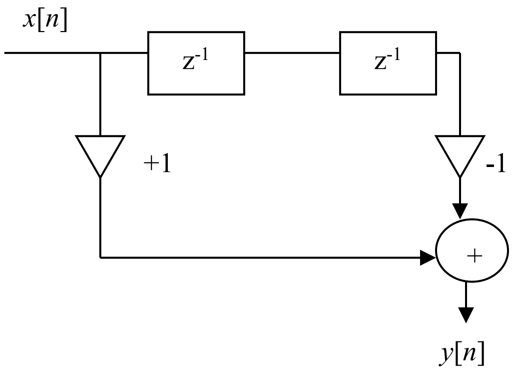
$$\boxed{\begin{aligned} \phi(\theta) &= -a\theta \\ \phi(\theta) &= b - a\theta \end{aligned} \qquad (5.4)}$$

where ‘*a*’ and ‘*b*’ are constants.

Example:

Two filter structures are shown below. Show that both filters have linear phase.





$$y[n] = x[n] + x[n-1] + x[n-2]$$

$$H(z) = 1 + z^{-1} + z^{-2}$$

$$\begin{aligned} H(\theta) &= 1 + e^{-j\theta} + e^{-j2\theta} \\ &= e^{-j\theta}\left(e^{j\theta} + e^{-j\theta} + 1\right) \\ &= e^{-j\theta}(1 + 2\cos\theta) \end{aligned}$$

phase: $\phi(\theta) = -\theta$

linear phase

$$y[n] = x[n] - x[n-2]$$

$$H(z) = 1 - z^{-2}$$

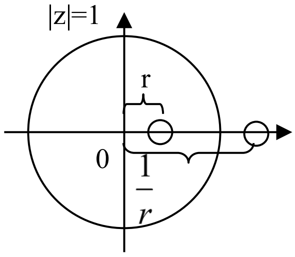
$$\begin{aligned} H(\theta) &= 1 - e^{-j2\theta} \\ &= 2je^{-j\theta}\left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right) \\ &= 2e^{j\frac{\pi}{2}}e^{-j\theta}\sin\theta \\ &= e^{j\left(\frac{\pi}{2}-\theta\right)}(2\sin\theta) \end{aligned}$$

phase: $\phi(\theta) = \pi/2 - \theta$

linear phase

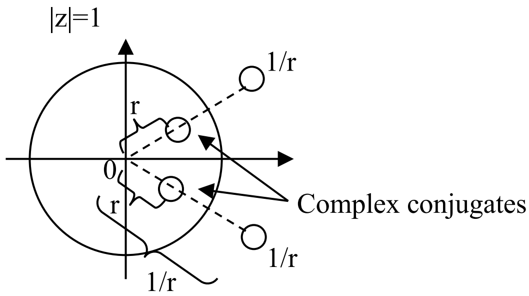
Pole-zero Patterns of Linear Phase Filters

- Linear phase filters provide constant delay with no amplitude distortion.
- An FIR filter with its impulse response $h[n]$ symmetric about the midpoint is endowed with linear phase and constant delay.
- For a linear phase FIR filter, the poles must lie at the origin (z=0).
- Impulse responses that are symmetric about the origin (i.e. n=0) require $h[n] = \pm h[-n]$ and hence $H(z) = \pm H\left(\frac{1}{z}\right)$
- The zeros of a linear-phase filter must occur in reciprocal pairs and exhibit conjugate reciprocal symmetry as shown below.



For a real zero

A zero on the real axis is paired with just its reciprocal



For a complex zero

Each zero not on the real axis is paired with its reciprocal and its conjugate.

- Zeros at z=1 or at z=-1 can occur singly, because they form their own reciprocal and their own conjugate.

- If there are no zeros at z=1, a linear-phase impulse response is always even symmetric about its midpoint.

- For odd symmetry about the mid-point, there must be an odd number of zeros at z=1.

- The frequency response of linear phase filter may be written as $H(\theta) = e^{j\alpha\theta} A(\theta)$ (for even symmetry) or $H(\theta) = je^{j\alpha\theta} A(\theta) = e^{j\left(\alpha\theta + \frac{\pi}{2}\right)} A(\theta)$ (for odd symmetry).

Example:

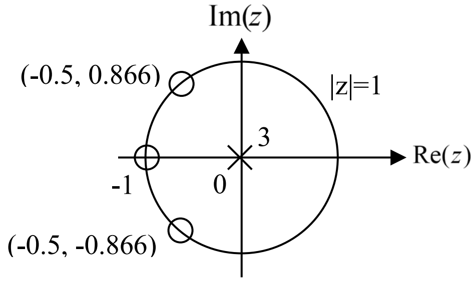
Is this a linear phase filter? Sketch the pole-zero plot.

$$H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

n=0 n=1 n=2 n=3

$$\therefore h[n] = \{1, 2, 2, 1\}$$

$$H(z) = \frac{z^3 + 2z^2 + 2z + 1}{z^3}$$



All of its poles are at z=0;

Its zeros are at z=-1 and z=-0.5±j0.866

The real zero at z=-1 can occur singly

Complex conjugate pair of zeros lie on the unit circle

<u>Ans: Linear phase filter, with an even symmetry impulse response *h[n]* about n=1.5 (mid point)</u>

Example:

$$H(z) = z^2 + 4.25 + z^{-2}$$

$$\therefore h[n] = \{1, 0, 4.25, 0, 1\}$$

n=-2 n=-1 n=0 n=1 n=2

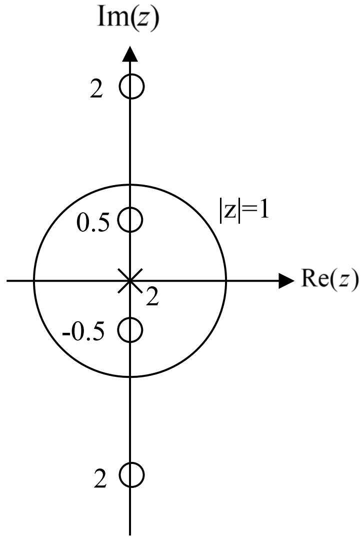
Is $H(z)$ linear phase? Sketch the pole-zero plot of $H(z)$.

Since $h[n]$ is even symmetric about n=0 with $h[n]$=$h[-n]$, we have $H(z)$=$H(1/z)$

$$H(z) = z^2 + 4.25 + z^{-2}$$

$$= \frac{z^4 + 4.25z^2 + 1}{z^2}$$

$$= \frac{(z + j0.5)(z - j0.5)(z + j2)(z - j2)}{z^2}$$



The four zeros at z = j0.5, z = j2, z = -j0.5 and z = -j2 are conjugate receiprocal symmetry. All poles are at z = 0.

Ans: Linear phase filter

Exercise:

H(z) = 1-2z^{-1}+z^{-2}

Is H(z) linear phase? Sketch the pole-zero plot of H(z).

Four types of linear phase filters:

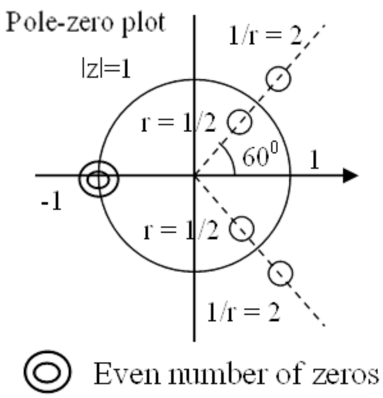
Type 1: Impulse response has even symmetry and odd length [Must have even number of zeros at z=-1 (if present) and z=1 (if present)]

Type 2: Impulse response has even symmetry and even length [Must have an odd number of zeros at z=-1 (if present) and even number of zeros at z=1 (if present)]

Type 3: Impulse response has odd symmetry and odd length. [Must have an odd number of zeros at z=-1 and odd number of zeros at z=1]

Type 4: Impulse response has odd symmetry and even length [Must have an odd number of zeros at z=1. The number of other zeros, if present at z=-1, must be even]

Example: Find all of the zero locations of a type 1 linear-phase sequence if it is known that there is a zero at $z=\frac{1}{2}e^{j\frac{\pi}{3}}$ and a zero at $z=-1$.



Due to conjugate reciprocal symmetry, the zero at $z=\frac{1}{2}e^{j\frac{\pi}{3}}$ implies we have a zero at $z=2e^{j\frac{\pi}{3}}$, $z=\frac{1}{2}e^{-j\frac{\pi}{3}}$ and $z=2e^{-j\frac{\pi}{3}}$. For a type 1 sequence, the number of zeros at z=-1 must be even, so there must be another zero at z=-1. Thus, there are 6 zeros.

5.1.2 Recursive digital filter (IIR)

Every recursive digital filter must contain at least one closed loop. Each closed loop contains at least one delay element.

$$y[n]=\sum_{k=0}^{M} a_k x[n-k]-\sum_{k=1}^{L} b_k y[n-k]$$

For recursive digital filters $b_k \neq 0$. Let $a_0 = a_0$, $a_k = 0$ for $k > 1$ and $b_1 = b_1$ & $b_k = 0$ for $k > 1$.

$$y[n] = a_0 x[n]-b_1 y[n-1]$$

$$H(z)=\frac{a_0}{1+b_1 z^{-1}}$$

IIR filter

A recursive filter is an infinite impulse response filter (IIR).

Example: 2nd order FIR filter

Transfer function	Description
$H_1(z)=a_0+a_1 z^{-1}+a_2 z^{-2}$	all-zero second order FIR filter (excluding poles at origin)
$H_2(z)=\dfrac{1}{1+b_1 z^{-1}+b_2 z^{-2}}$	all-pole second order IIR filter (excluding zeros at origin)
$H_3(z)=\dfrac{1+a_1 z^{-1}+a_2 z^{-2}}{1+b_1 z^{-1}+b_2 z^{-2}}$	pole-zero second order IIR filter

Note: Poles and zeros can be real or imaginary

Example: Consider the 1st order IIR filter

$$H(z) = \frac{1}{1 - az^{-1}} \qquad 0 < a < 1$$

The difference equation is: $y[n] = x[n] + ay[n\text{-}1]$.

The DC gain of $H(z)$ can be obtained by substituting $\theta = 0$.

$$H(\theta)|_{\theta=0} = \frac{1}{1 - ae^{-j(0)}} = \frac{1}{1-a}$$

If DC gain is undesirable, introduce a constant gain factor of (1-a), so that $H(z)$ becomes

$$H(z) = \frac{1-a}{1 - az^{-1}} \qquad \text{DC gain} = 1$$

$$\therefore\ y[n] = (1\text{-}a)x[n] + ay[n\text{-}1]$$

Example: Consider a lowpass filter

$$y[n] = ay[n\text{-}1] + bx[n], \qquad 0 < a < 1$$

(i) Determine 'b' so that $|H(0)| = 1$.
(ii) Determine the 3dB bandwidth (here) for the normalised filter in part (i).

(i) $Y(z) = aY(z)\, z^{-1} + b\, X(z) \Rightarrow H(z) = \dfrac{b}{1 - az^{-1}}$

$H(z) = \dfrac{b}{1 - ae^{-j\theta}}$ we have $|H(0)| = 1$

$$H(0) = \frac{b}{1-a} = 1 \Rightarrow b = 1 - a$$

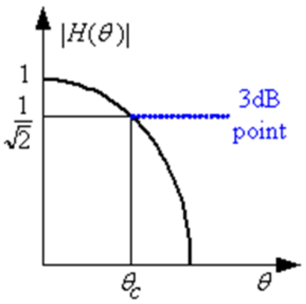
$$H(\theta)=\frac{1-a}{1-ae^{-j\theta}}=\frac{1-a}{(1-a\cos\theta)+ja\sin\theta}$$

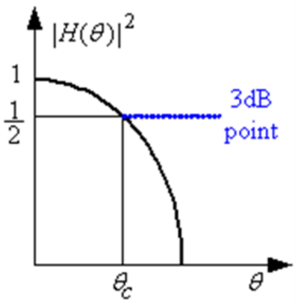
$$|H(\theta)|=\frac{1-a}{\sqrt{(1-a\cos\theta)^2+(a\sin\theta)^2}}=\frac{1-a}{\sqrt{1+a^2-2a\cos\theta}}$$

Second Method:

$$\left\{|H(\theta)|^2=H(\theta)\cdot H^*(\theta)=\frac{1-a}{1-ae^{-j\theta}}\cdot\frac{1-a}{1-ae^{j\theta}}=\frac{(1-a)^2}{1-2a\cos\theta+a^2}\right\}$$

$$\therefore\ |H(\theta)|=\frac{1-a}{\sqrt{1-2a\cos\theta+a^2}} \qquad |H(0)|^2=1\ \therefore |H(\theta)|^2_{\theta=\theta_c}=\frac{1}{2}|H(0)|^2$$





$$\frac{1}{2}=\frac{(1-a)^2}{1+a^2-2a\cos\theta_c}$$

$$\theta_c=\cos^{-1}\left[\frac{-a^2+4a-1}{2a}\right]$$

<u>Example:</u> Consider a filter described by

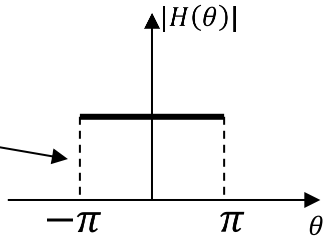
$$H(z)=\frac{a+cz^2}{c+az^2}\quad \text{where } a \ \&\ c \text{ are constants.}$$

Show that the magnitude response $|H(\theta)|$ is unity for all θ.

$$|H(\theta)|^2=H(\theta)\cdot H^*(\theta)=\frac{c+ae^{-j2\theta}}{a+ce^{-j2\theta}}\cdot\frac{c+ae^{j2\theta}}{a+ce^{j2\theta}}$$

$$=\frac{c^2+a^2+ac[e^{j2\theta}+e^{-j2\theta}]}{a^2+c^2+ac[e^{j2\theta}+e^{-j2\theta}]}=1$$

This is an **all-pass** filter.



Exercise: Determine the magnitude response of the following filter and show that it has an all-pass characteristic.

$$H(z) = \frac{1-(1/a)z^{-1}}{((1/a)-z^{-1})} \qquad |a| < 1$$

5.2 Digital Filter Realisation

$$y[n] = \sum_{k=0}^{M} a_k x[n-k] - \sum_{k=1}^{L} b_k y[n-k]$$

$$Y(z) = \sum_{k=0}^{M} a_k X(z) z^{-k} - \sum_{k=1}^{L} b_k Y(z) z^{-k}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{1 + \sum_{k=1}^{L} b_k z^{-k}}$$

Structure I : $$H(z) = \underbrace{\sum_{k=0}^{M} a_k z^{-k}}_{zeros} \cdot \frac{1}{\underbrace{1 + \sum_{k=1}^{L} b_k z^{-k}}_{poles}}$$

Structure II : $$H(z) = \frac{1}{\underbrace{1 + \sum_{k=1}^{L} b_k z^{-k}}_{poles}} \cdot \underbrace{\sum_{k=0}^{M} a_k z^{-k}}_{zeros}$$

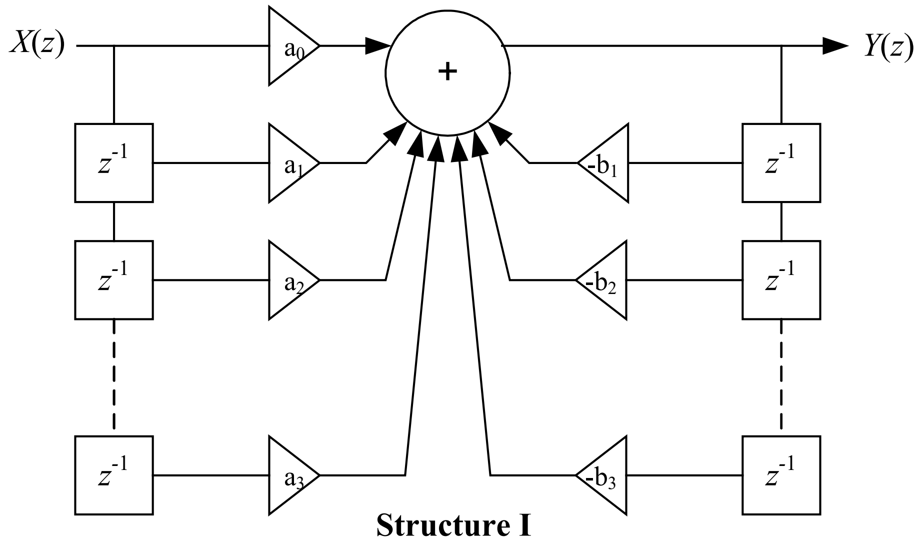


Figure 5.1 Structure I or Direct Form I

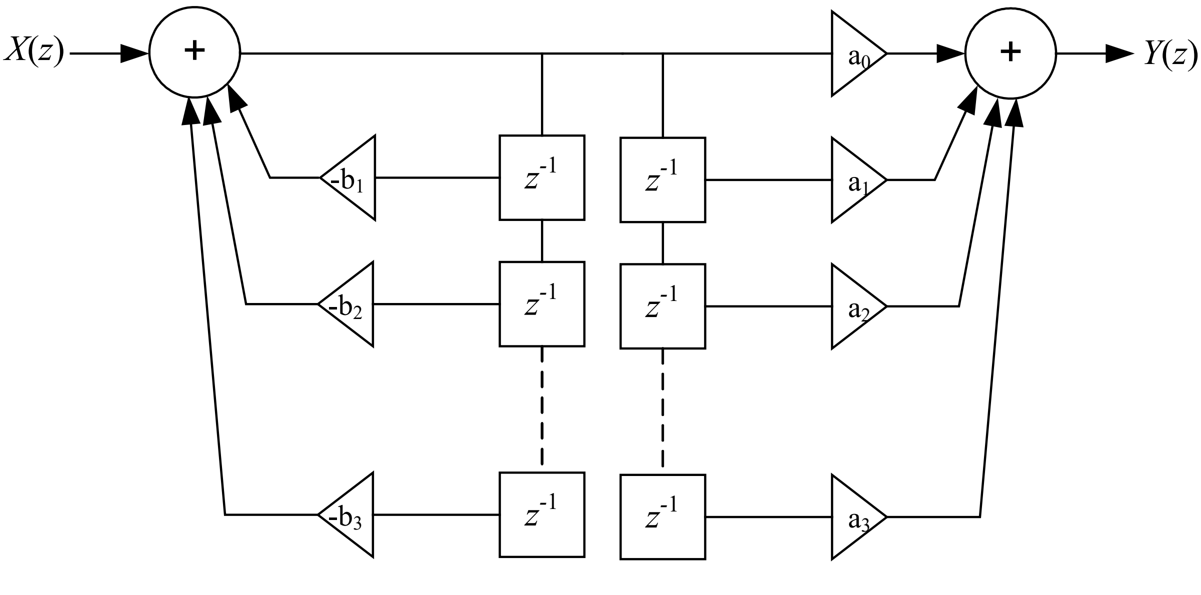


Figure 5.2 Structure II or Direct Form II

In the case when $L = M$, we have canonic form realisation.

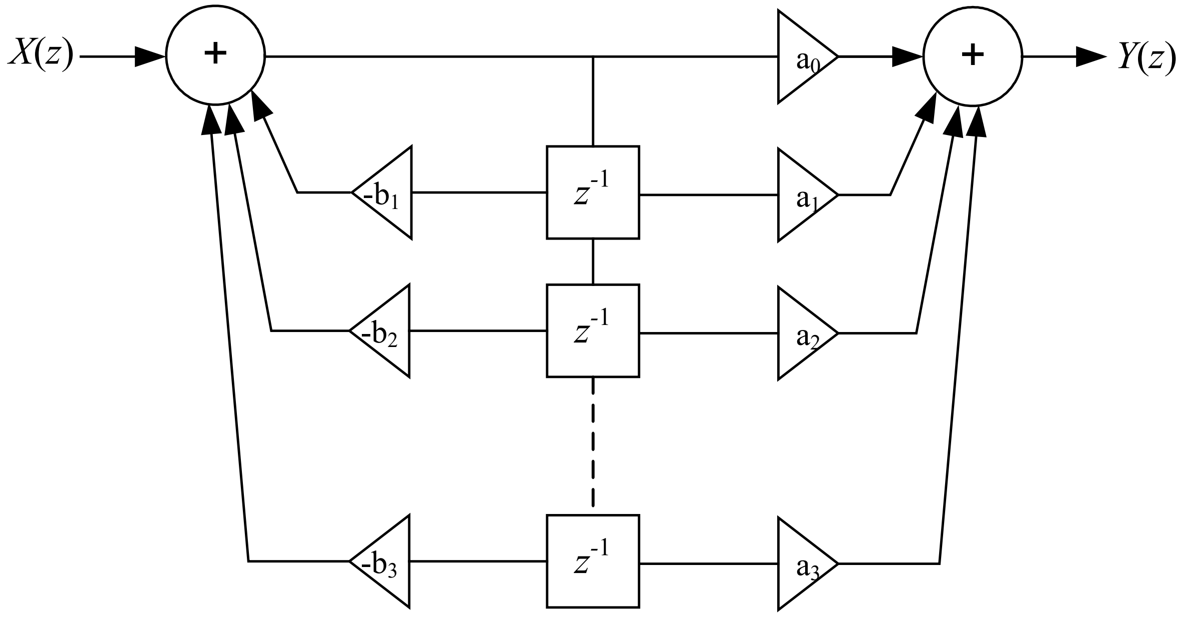


***Figure 5.3**: Canonic form*

A discrete-time filter is said to be canonic if it contains the minimum numbers of delay elements necessary to realise the associated frequency response.

5.2.1 Parallel realisation

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_L z^{-L}}$$

$$= \underbrace{\sum_{i=1}^{k} H_i(z) = H_1(z) + H_2(z) + H_3(z) + \ldots + H_k(z)}_{parallel\_structure}$$

(use partial fraction to obtain $H_i(z)$)

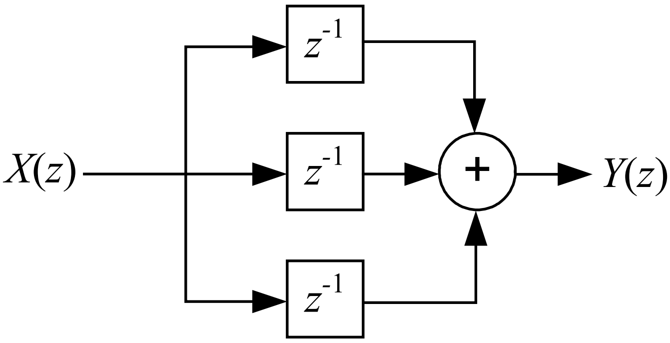


Figure 5.4 Parallel structure

<u>Example</u>: Draw a parallel realisation of a third order system $H(z)$ as given below:

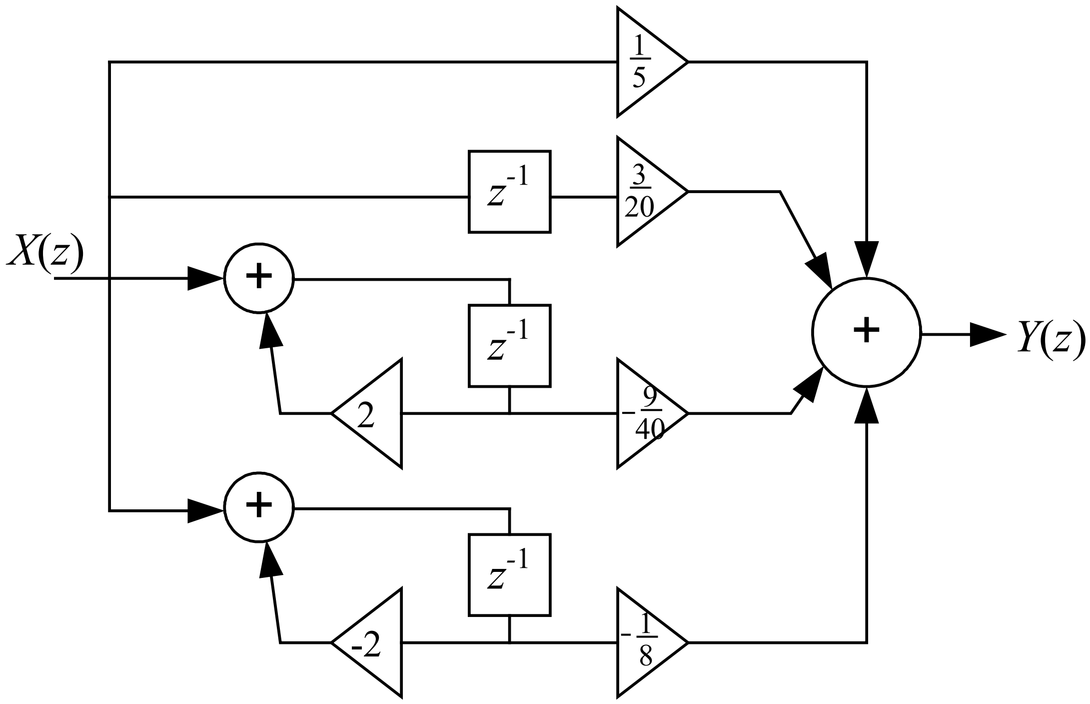
$$H(z) = \frac{z^3 - z^2 - 5z - 3}{5z(z^2 - 4)}$$

By using a partial fraction decomposition, $H(z)$ becomes:

$$H(z) = A + \frac{B}{z} + \frac{C}{z-2} + \frac{D}{z+2}$$

$$H(z) = \frac{1}{5} + \frac{3}{20z} - \frac{9}{40(z-2)} - \frac{1}{8(z+2)}$$

$$H(z) = \frac{1}{5} + \frac{3}{20}z^{-1} - \frac{9z^{-1}}{40(1-2z^{-1})} - \frac{z^{-1}}{8(1+2z^{-1})}$$



5.2.2 Cascade realisation

$$H(z)=\frac{a_0+a_1z^{-1}+a_2z^{-2}+\cdots+a_Mz^{-M}}{1+b_1z^{-1}+b_2z^{-2}+\cdots+b_Lz^{-L}}$$

$$H(z)=\prod_{i=1}^{k}\hat{H}_i(z)=\underbrace{\hat{H}_1(z)\cdot\hat{H}_2(z)\cdot\hat{H}_3(z)...\hat{H}_k(z)}_{cascade\_structure}$$

(Product of lower order transfer function ie. 1st or 2nd order sections). The cascade structure is the most popular form

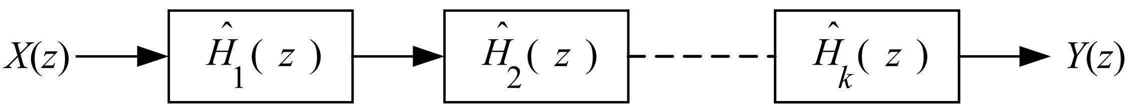


Figure 5.6: Cascade structure

<u>Example:</u> A parallel realisation of a third order system $H(z)$ is given by

$$H(z)=\frac{23+40z^{-1}+36z^{-2}+19z^{-3}}{(2+z^{-1})(5+2z^{-1}+3z^{-2})}$$

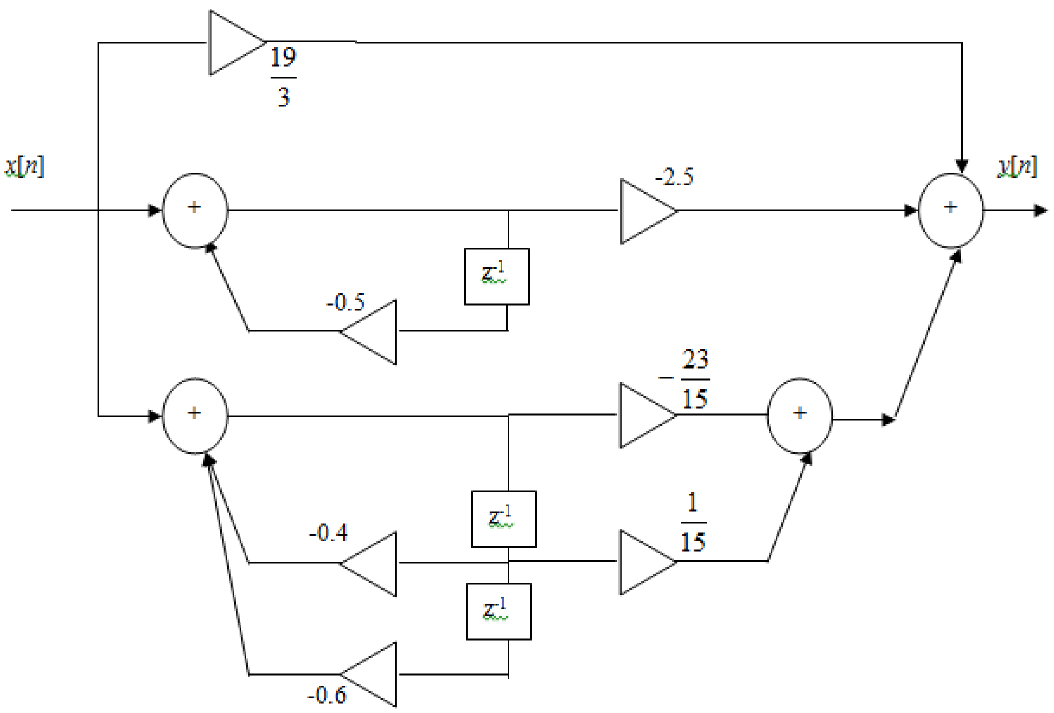
$$=A+\frac{B}{2+z^{-1}}+\frac{C-Dz^{-1}}{5+2z^{-1}+3z^{-2}}$$

$H_0(z)$ $H_1(z)$ $H_2(z)$

$$H(z)=\frac{19}{3}-\frac{5}{2+z^{-1}}-\frac{1}{3}\frac{(23-z^{-1})}{5+2z^{-1}+3z^{-2}}$$

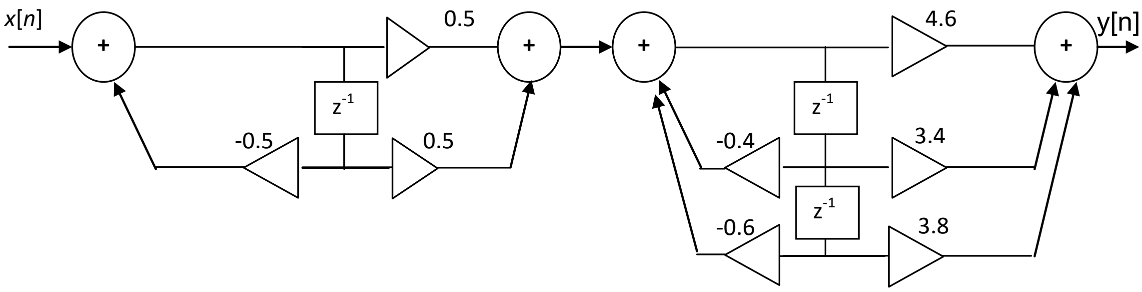
$$H(z)=\frac{19}{3}-\frac{5}{2}\frac{1}{1+0.5z^{-1}}-\frac{1}{3}\frac{(23-z^{-1})}{5(1+0.4z^{-1}+0.6z^{-2})}$$

$$=\frac{19}{3}+\frac{(-2.5)}{1+0.5z^{-1}}+\frac{(-\frac{23}{15}+\frac{1}{15}z^{-1})}{1+0.4z^{-1}+0.6z^{-2}}$$



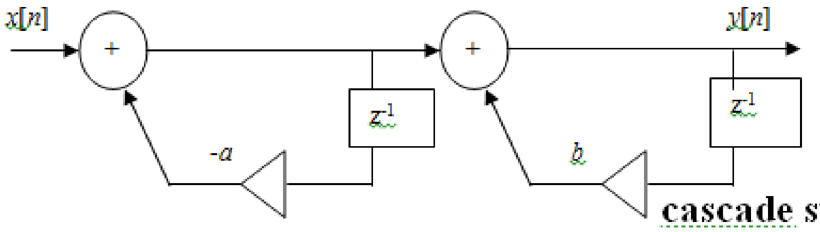
Example: A cascade realisation of a third-order system is given by

$$
\begin{aligned}
H(z) &= \frac{23+40z^{-1}+36z^{-2}+19z^{-3}}{10+9z^{-1}+8z^{-2}+3z^{-3}} \\
&= \frac{(1+z^{-1})}{(2+z^{-1})}\,\frac{23+17z^{-1}+19z^{-2}}{5+2z^{-1}+3z^{-2}} \\
&= \frac{(1+z^{-1})}{2(1+0.5z^{-1})}\,\frac{23+17z^{-1}+19z^{-2}}{5(1+0.4z^{-1}+0.6z^{-2})} \\
&= \frac{(0.5+0.5z^{-1})}{1+0.5z^{-1}}\,\frac{4.6+3.4z^{-1}+3.8z^{-2}}{1+0.4z^{-1}+0.6z^{-2}} \\
&= \left(\frac{(0.5+0.5z^{-1})}{1+0.5z^{-1}}\right)\left(\frac{4.6+3.4z^{-1}+3.8z^{-2}}{1+0.4z^{-1}+0.6z^{-2}}\right)
\end{aligned}
$$



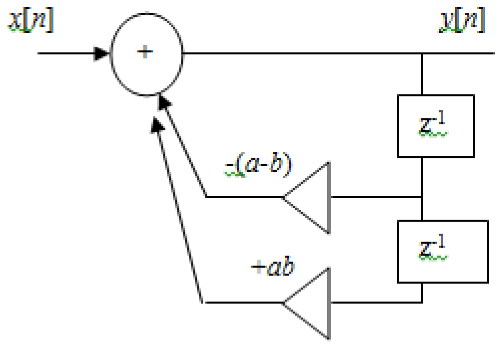
Example: Implement the following system in the cascade, direct form II and parallel structures. All coefficients are real.

(a) $H(z) = \dfrac{1}{(1+az^{-1})(1-bz^{-1})}$



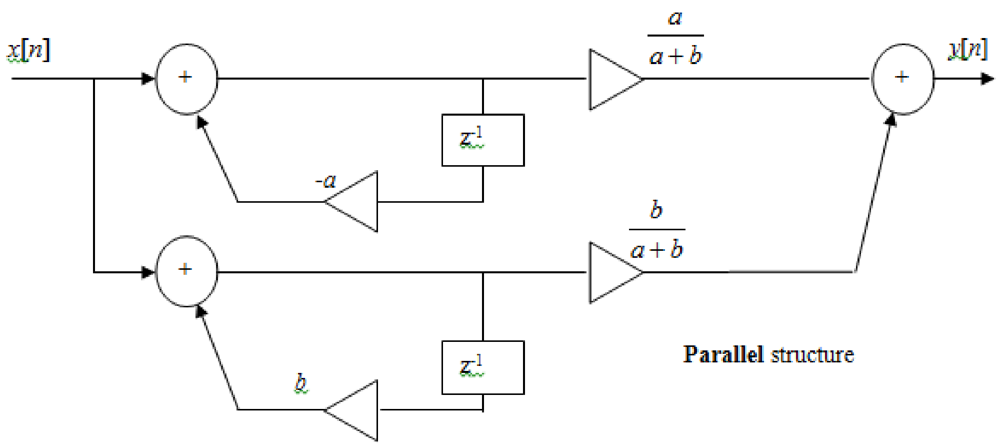
cascade structure

$$H(z)=\frac{1}{1+(a-b)z^{-1}-abz^{-2}}$$



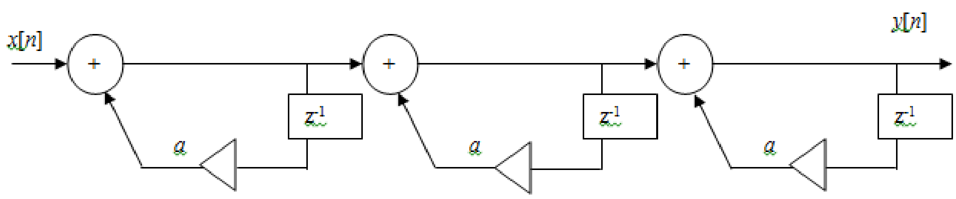
Direct form II

$$H(z)=\frac{A}{1+az^{-1}}+\frac{B}{1-bz^{-1}}=\frac{\frac{a}{a+b}}{1+az^{-1}}+\frac{\frac{b}{a+b}}{1-bz^{-1}}$$



Parallel structure

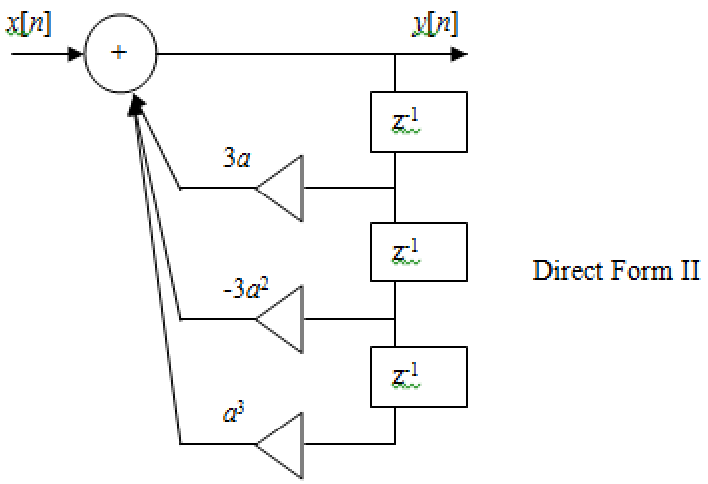
b) $H(z)=\frac{1}{(1-az^{-1})^3}$



cascade **structure**

No parallel structure exists because partial fraction expansion cannot be performed.

$$H(z) = \frac{1}{1 - 3az^{-1} + 3a^2z^{-2} - a^3z^{-3}}$$

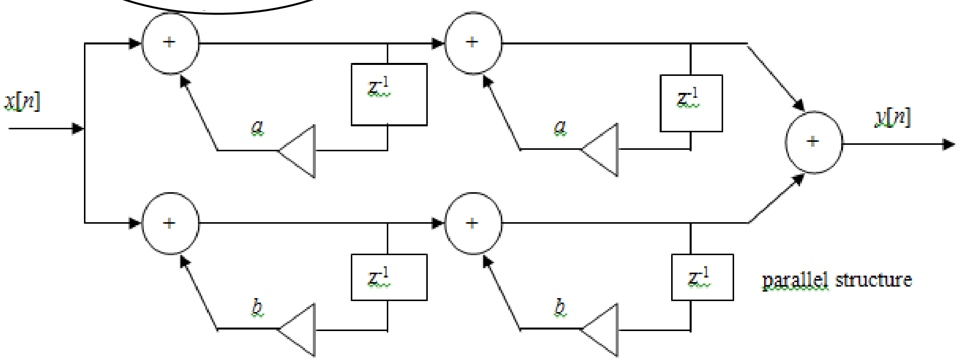


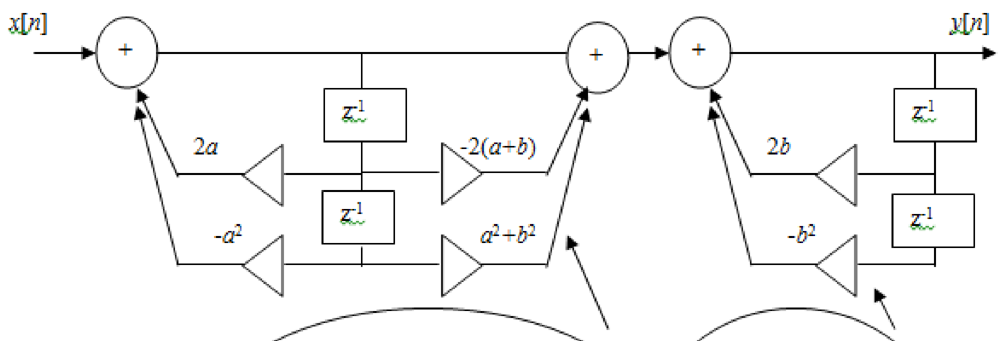
(c)

$$H(z) = \frac{1}{(1 - za^{-1})^2} + \frac{1}{(1 - bz^{-1})^2} \quad \leftarrow \quad parallel$$

$$= \frac{1 - 2(a+b)z^{-1} + (a^2 + b^2)z^{-1}}{(1 - 2az^{-1} + a^2z^{-2})(1 - 2bz^{-1} + b^2z^{-2})} \quad \leftarrow \quad cascade$$

$$H(z) = \frac{1}{(1 - za^{-1})(1 - za^{-1})} + \frac{1}{(1 - bz^{-1})(1 - bz^{-1})}$$





$$H(z) = \frac{1-2(a+b)z^{-1}+(a^2+b^2)z^{-2}}{(1-2az^{-1}+a^2z^{-2})} \cdot \frac{1}{1-2bz^{-1}+b^2z^{-2}}$$

Exercise:

A first-order digital filter has a transfer function given by:

$$H(z) = \frac{1}{4}\frac{1+z^{-1}}{1-0.5z^{-1}}$$

Draw the following realisations of H(z):
(a) Cascade; (b) Parallel; (c) Canonic.

5.3 Magnitude and Phase Response

- We can show that the magnitude response is an **even** function of frequency.
- The phase response is an **odd** function of frequency.

Example: Calculate the magnitude and phase response of the 3-sample averager given by

$$h[n] = \begin{cases} \frac{1}{3} & -1 \le n \le 1 \\ 0 & otherwise \end{cases}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \sum_{n=-1}^{1} h(n) z^{-n} = \frac{1}{3} z^{-1} + \frac{1}{3} z^{0} + \frac{1}{3} z^{1}$$

$$H(z) = \frac{1}{3}\left[z^{-1} + 1 + z^{1}\right]$$

$$H(\theta) = H(z)\big|_{z=e^{j\theta}} = \frac{1}{3}\left[e^{-j\theta} + 1 + e^{j\theta}\right] = \left[1 + e^{j\theta} + e^{-j\theta}\right] \cdot \frac{1}{3}$$

$$H(\theta) = \frac{1}{3}\left[1 + 2\cos\theta\right]$$

Precautions must be taken when determining the phase response of a filter having a <u>real-valued</u> transfer function H(θ), because negative real values produce an additional phase of π radians.

For example, let us consider the following <u>linear-phase</u> form of the transfer function

$$H(\theta) = e^{-jk\theta} B(\theta)$$

real-valued function of θ that can take positive and negative values.

$$H(\theta) = B(\theta)\cos(-k\theta) + B(\theta) j \sin(-k\theta)$$

$$H(\theta) = B(\theta)\cos(k\theta) - jB(\theta)\sin(k\theta)$$

Let phase angle be ϕ:

$$\tan\phi = -\frac{B(\theta)\sin(k\theta)}{B(\theta)\cos(k\theta)} = -\tan(k\theta)$$

$\tan\phi = \tan(-k\theta) \therefore \phi = -k\theta \quad$ or $\phi(\theta) = -k\theta$

phase angle

The phase function $\phi(\theta)$ includes linear phase term and also accommodates for the sign changes in $B(\theta)$. Since -1 can be expressed as $e^{\pm j\pi}$, phase jumps of $\pm\pi$ will occur at frequencies where $B(\theta)$ changes sign.

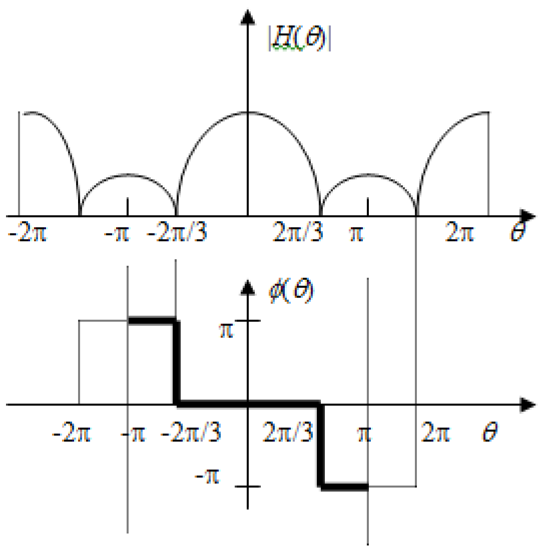
If $B(\theta) > 0$, then $\phi(\theta) = -k\theta$.
If $B(\theta) < 0$, then $\phi(\theta) = -k\theta \pm \pi$..

Let us get back to our example of the 3-sample averager

$$H(\theta) \quad = \frac{1}{3}[1+2\cos\theta] \Rightarrow |H(\theta)| = \left|\frac{1}{3}[1+2\cos\theta]\right|$$

$$\begin{cases} \phi(\theta) = 0 & H(\theta) > 0 & -\frac{2\pi}{3} < \theta < \frac{2\pi}{3} \\ \phi(\theta) = 0 \pm \pi & H(\theta) < 0 & -\pi \le \theta \le -\frac{2\pi}{3} \text{ and } \frac{2\pi}{3} < \theta < \pi \end{cases}$$

The appropriate sign of π must be chosen to make $\phi(\theta)$ an odd function of frequency.



Even function

Odd function

Example: Find the magnitude and phase response of the following: $h(0)=\frac{1}{2},\quad h(1)=1,\quad h(2)=\frac{1}{2},\quad h(n)=0$, otherwise.

$$H(z)=\frac{1}{2}z^{0}+1z^{-1}+\frac{1}{2}z^{-2}=\frac{1}{2}+z^{-1}+\frac{1}{2}z^{-2}$$

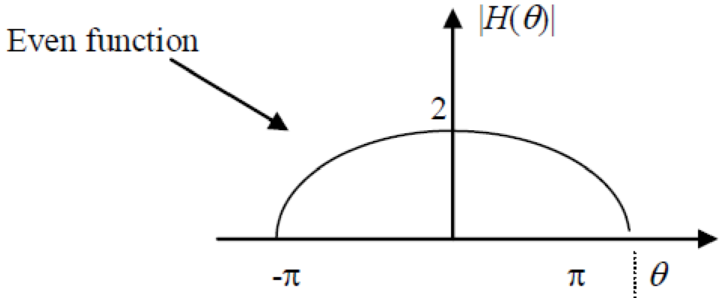
$$H(\theta)=\frac{1}{2}+e^{-j\theta}+\frac{1}{2}e^{-j2\theta}$$

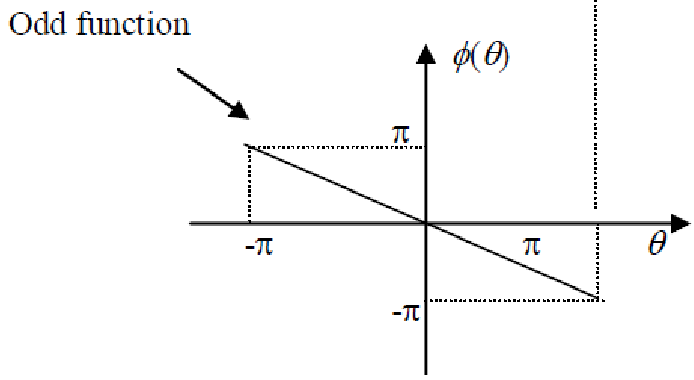
$$=e^{-j\theta}\left[\frac{1}{2}e^{j\theta}+1+\frac{1}{2}e^{-j\theta}\right]$$

$$H(\theta)=e^{-j\theta}\underbrace{[1+\cos\theta]}_{B(\theta)}$$

$$\phi(\theta)=-\theta$$

The amplitude function is never negative (therefore there is no phase jumps of $\pm\pi$)

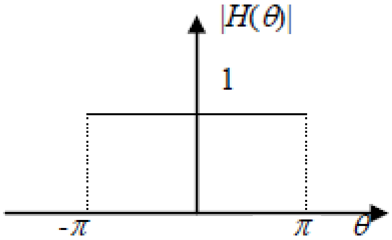


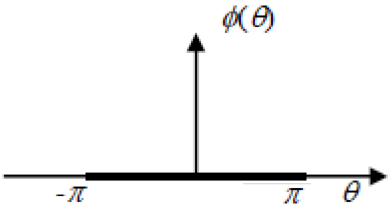


Example:

Case 1:

$$\left.\begin{array}{ll}\delta(n) = 1 & n = 0 \\ \delta(n) = 0 & otherwise\end{array}\right\} \quad case1$$



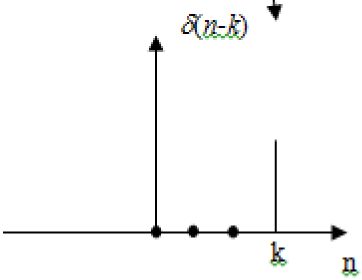


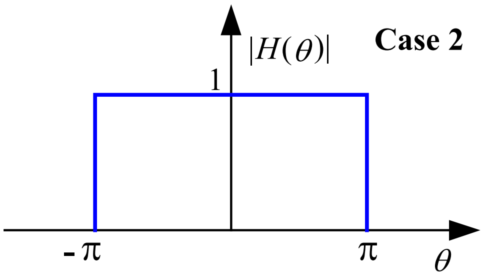
Case 2:

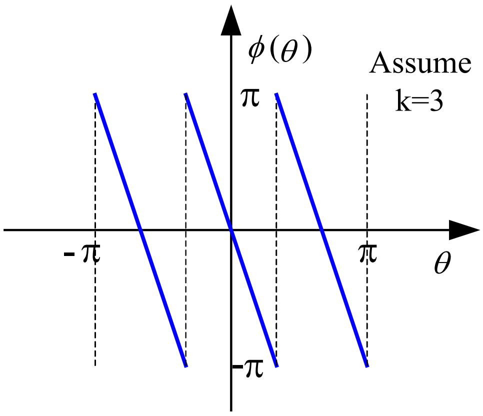
$h[n] = \delta[n\text{-}k]$

$H(z) = 1z^{-k}$

$H(\theta) = 1e^{-j\theta k}$







$B(\theta) = 1$

$\phi(\theta) = -\theta k$

Note: When phase exceeds ±π range a jump of ±2π is needed to bring the phase back into ±π range.

Phase Jumps: From the previous examples, we note that there are two occasions for which the phase function experiences discontinuities or jumps.

1. A jump of ±2π occurs to maintain the phase function within the principal value range of [-π and π]
2. A jump of ± π occurs when $B(\theta)$ undergoes a change of sign

The sign of the phase jump is chosen such that the resulting phase function is odd and, after the jump, lies in the range [-π and π].

Example: Magnitude and phase response of causal 3-sample average.

$$h[n] = \begin{cases} \frac{1}{3} & \text{for} \quad 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$H(z) = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2} \Rightarrow H(\theta) = \frac{1}{3} + \frac{1}{3}e^{-j\theta} + \frac{1}{3}e^{-j2\theta}$$

$$= \frac{1}{3}e^{-j\theta}\left[e^{j\theta} + 1 + e^{-j\theta}\right]$$

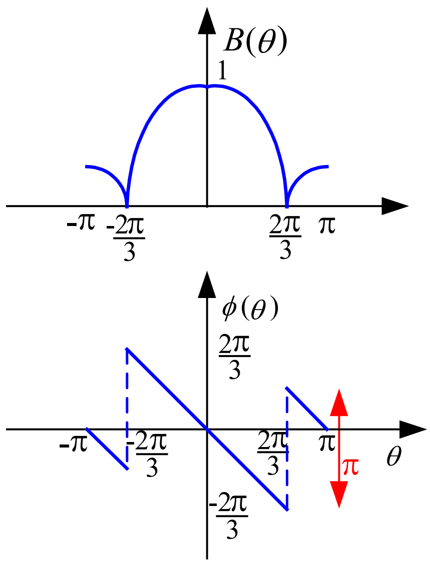
$$\therefore H(\theta) = e^{-j\theta}\underbrace{\frac{1}{3}[1 + 2\cos\theta]}_{B(\theta)}$$

$$\phi(\theta) = -\theta; \qquad |H(\theta)| = |B(\theta)| = \frac{1}{3}[1 + 2\cos\theta]$$

$$\phi(\theta) = -\theta \qquad B(\theta) > 0 \qquad -\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$$

$$= -\theta \pm \pi \qquad B(\theta) < 0 \qquad -\pi < \theta < -\frac{2\pi}{3}$$

$$\frac{2\pi}{3} < \theta < \pi$$



Phase is undefined at points $|H(\theta)| = 0$ or $B(\theta)=0$.

Example: Determine and sketch the magnitude and phase response of 1st order recursive filter (IIR filter)

$$y[n] = x[n] + ay[n-1]$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1-az^{-1}} \qquad\qquad H(\theta) = \frac{1}{1-ae^{-j\theta}}$$

$$H(\theta) = \frac{1}{1-ae^{-j\theta}} \cdot \frac{1-ae^{j\theta}}{1-ae^{j\theta}} = \frac{1-a\cos\theta}{1-2a\cos\theta+a^2} - \frac{ja\sin\theta}{1-2a\cos\theta+a^2}$$

$$\tan\phi = \frac{-\dfrac{a\sin\theta}{1-2a\cos\theta+a^2}}{\dfrac{1-a\cos\theta}{1-2a\cos\theta+a^2}} = \frac{-a\sin\theta}{1-a\cos\theta}$$

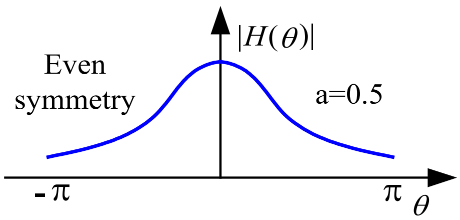
$$\phi(\theta) = \tan^{-1}\left[\frac{-a\sin\theta}{1-a\cos\theta}\right] \qquad \textit{phase}$$

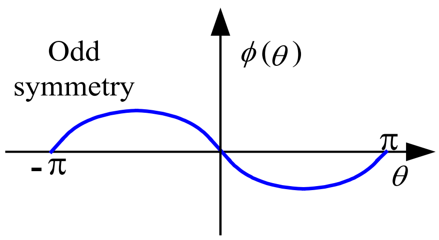
Magnitude:

$|H(\theta)|^2 = H(\theta)\cdot H^*(\theta)$ ($H^*(\theta)$ is the complex conjugate of $H(\theta)$)

$$= \frac{1}{1-ae^{-j\theta}} \cdot \frac{1}{1-ae^{j\theta}} \qquad 0 < a < 1$$

$$\boxed{|H(\theta)| = \frac{1}{1-2a\cos\theta+a^2}}$$





Example:

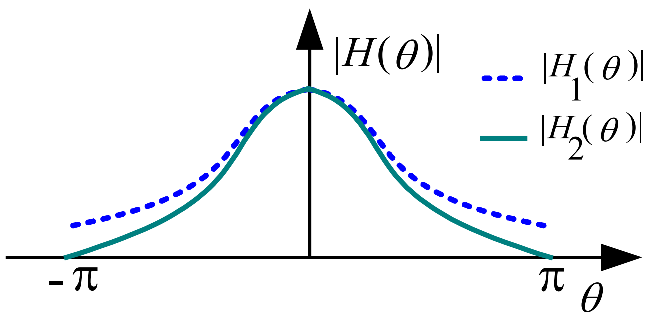
(a) $H_1(z) = \frac{k_0}{1 - az^{-1}} \Rightarrow \frac{1-a}{1-az^{-1}}$ (Low pass filter)

The gain k_0 can be selected as $1 - a$, so that the filter has unity gain at $\theta = 0$.

(b) $H_2(z) = k_0 \frac{1 + z^{-1}}{1 - az^{-1}}$

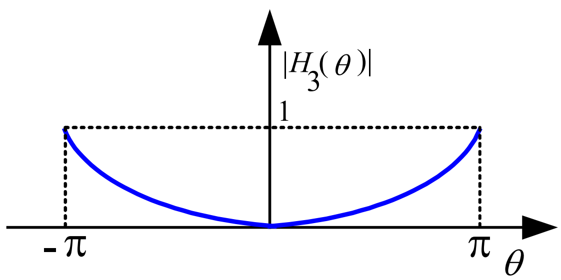
In this case, $k_0 = \frac{1-a}{2}$ (for unity gain at $\theta = 0$).

The addition of a zero at z = -1 further attenuates the response of the filter at high frequencies



(c) We can obtain simple highpass filters by reflecting (folding) the pole-zero locations of the lowpass filters about the imaginary axis in the z-plane.

$H_3(z) = \frac{1-a}{2} \cdot \frac{1 - z^{-1}}{1 + az^{-1}}$ high pass filter

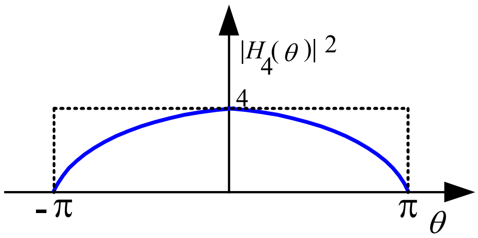


(d)

$$H_4(z) = 1 + z^{-1}$$

$$y[n] = x[n] + x[n-1]$$ lowpass filter

$|H_4(\theta)|^2 = 2(1+\cos\theta)$

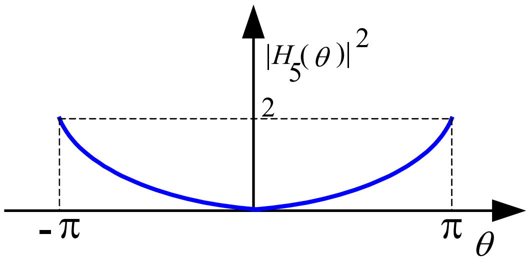


(e)

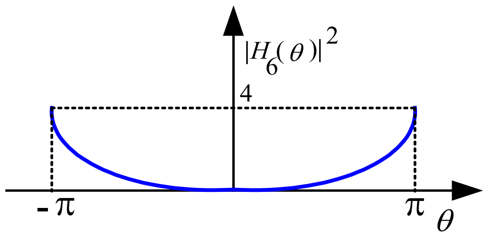
$$H_5(z) = 1 - z^{-1}$$

$$y[n] = x[n] - x[n-1]$$ high pass filter

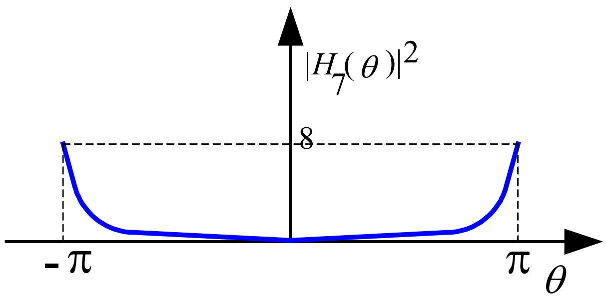
$|H_5(\theta)|^2 = 2(1-\cos\theta)$



(f) $H_6(z) = \left(1 - z^{-1}\right)^2$



(g) $H_7(z) = \left(1 - z^{-1}\right)^3$



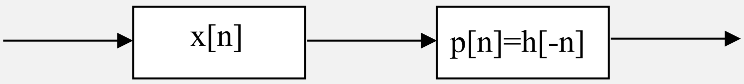
<u>Exercise</u>:

(a) Calculate and Sketch the Magnitude and Phase responses of the following FIR filter

$$y[n] = \tfrac{1}{4}\big[x[n] - 2x[n-1] + x[n-2]\big]$$

(b) Show that if x[n] is a real valued signal, the structure shown below has a zero-phase response; where h[n] is the impulse response of the filter H(z) given by,

$$H(z) = (\frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2})$$

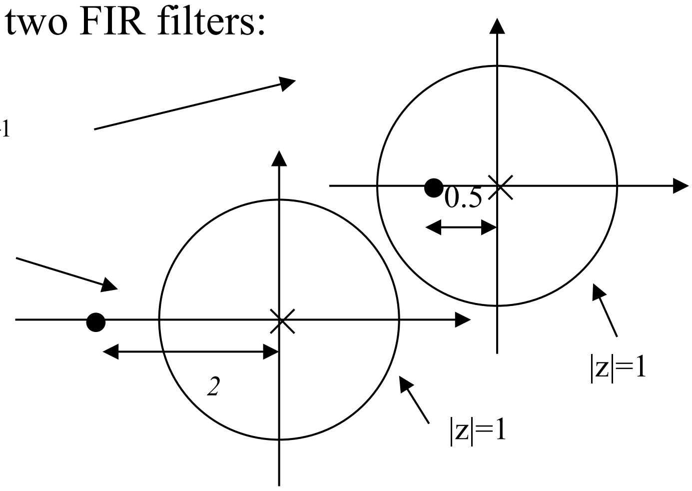


5.4 Minimum, Maximum and Mixed phase systems

Let us consider two FIR filters:

$$H_1(z) = 1 + \frac{1}{2}z^{-1}$$

$$H_2(z) = \frac{1}{2} + z^{-1}$$



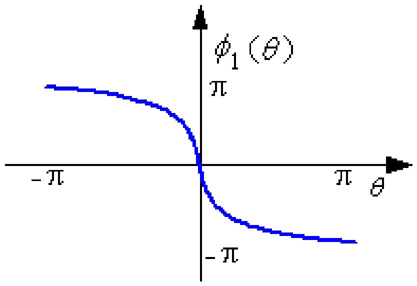
$H_2(z)$ is the reverse of the system $H_1(z)$. This is due to the reciprocal relationship between the zeros of $H_1(z)$ and $H_2(z)$.

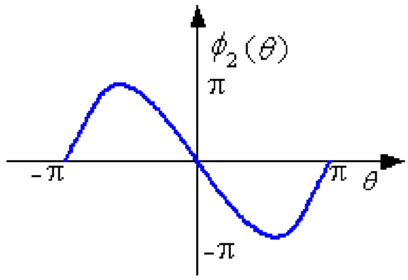
$$H_1(\theta) = 1 + \frac{1}{2}e^{-j\theta} \;\&\; H_2(\theta) = \frac{1}{2} + e^{-j\theta}$$

$$|H_1(\theta)| = |H_2(\theta)| = \sqrt{\frac{5}{4} + \cos\theta}$$

The <u>magnitude</u> characteristics for the two filters are identical because the roots of $H_1(z)$ and $H_2(z)$ are reciprocal.

$$\phi_1(\theta) = \tan^{-1}\frac{-\sin\theta}{2+\cos\theta} \qquad \phi_2(\theta) = \tan^{-1}\frac{-\sin\theta}{\frac{1}{2}+\cos\theta}$$





Note: If we reflect a zero with magnitude $|z| = \rho$ that is inside the unit circle into a zero with magnitude $|z| = \frac{1}{\rho}$ outside the unit circle the magnitude characteristic of the system is unaltered, but the phase response changes.

We observe that the phase characters $\phi_1(\theta)$ begins at zero phase at frequency $\theta = 0$ and terminates at zero phase at the frequency $\omega = \pi$. Hence the net phase change.

$$\boxed{\phi_1(\pi) - \phi_1(0) = 0}$$ Minimum phase filter

On the other hand, the phase characteristic for the filter with the zero outside the unit circle undergoes a net phase change

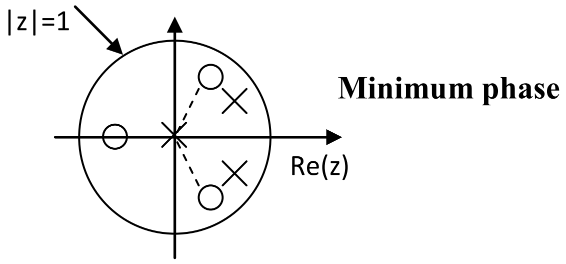
$$\boxed{\phi_2(\pi) - \phi_2(0) = -\pi \ \text{ radians}}$$

As a consequence of these different phase characteristics, we call the first filter a **minimum-phase** system and the second system is called a **maximum-phase** system.

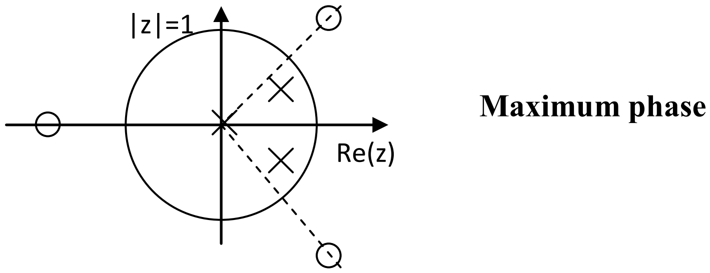
If a filter with M zeros has some of its zeros inside the unit circle and the remaining outside the unit circle, it is called a **mixed-phase** system. A minimum-phase property of FIR filter carries over to IIR filter. Let us consider

$$H(z) = \frac{B(z)}{A(z)}$$

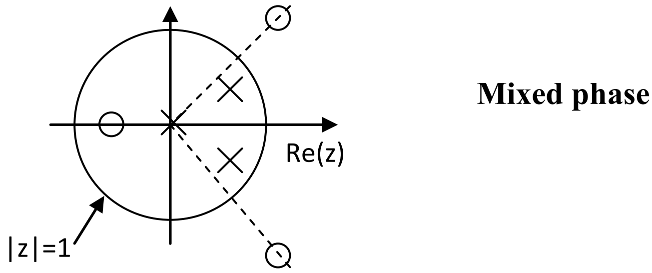
is called minimum phase if all its poles and zeros are inside the unit circle.



If all the zeros lie outside the unit circle, the system is called maximum phase.

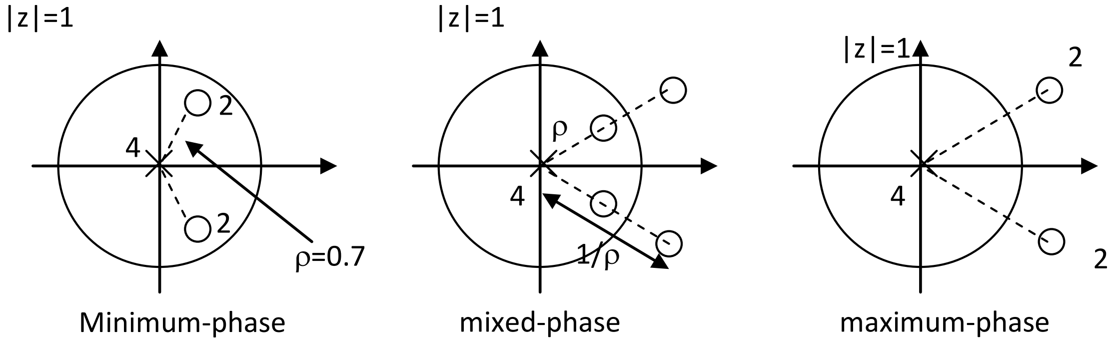


If zeros lie both inside and outside the unit circle, the system is called mixed-phase.

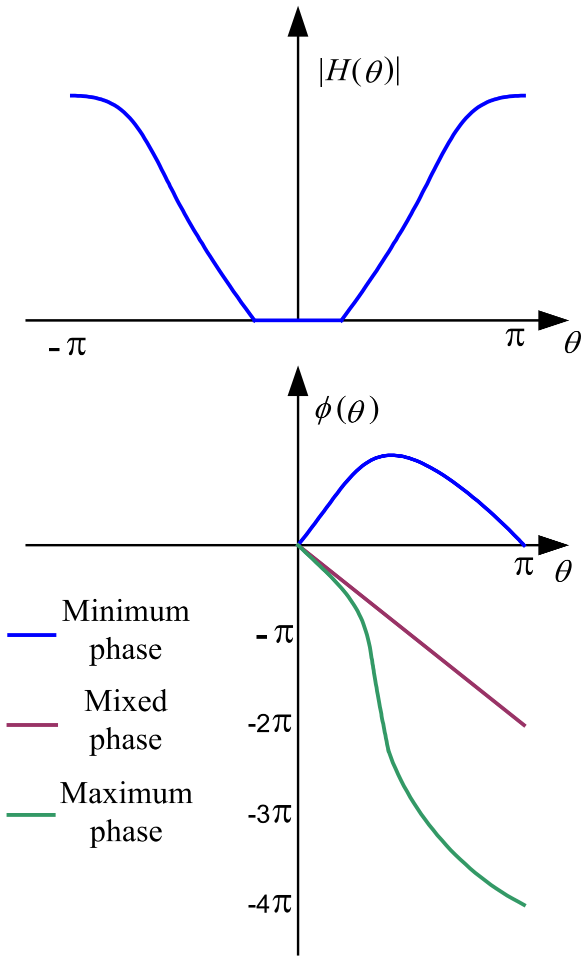


Note: For a given magnitude response, the minimum-phase system is the causal system that has the smallest magnitude phase at every frequency (θ). That is, in the set of causal and stable filters having the same magnitude response, the minimum-phase response exhibits the smallest deviation from zero phase.

Example: Consider a fourth-order all-zero filter containing a double complex conjugate set of zeros located at $z = 0.7e^{\pm j\frac{\pi}{4}}$. The minimum-phase, mixed phase and maximum phase system pole-zero patterns having identical magnitude response are shown below.



The magnitude response and the phase response of the three systems are shown below: The minimum-phase system seems to have the phase with the smallest deviation from zero at each frequency.



*Mixed-phase is linear in this case

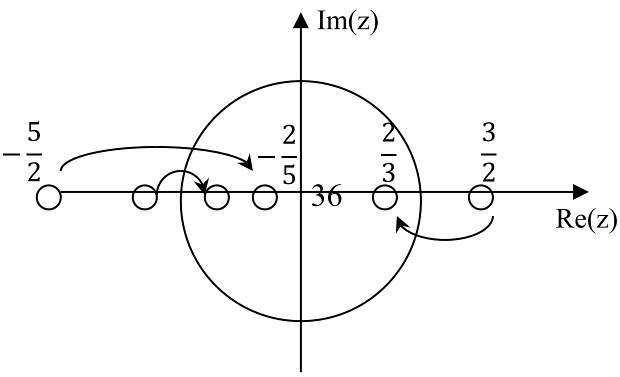
Example: A third order FIR filter has a transfer function $G(z)$ given by: $G(z) = (6 - z^{-1} - 12z^{-2})(2 + 5z^{-1})$

From $G(z)$, determine the transfer function of an FIR filter whose magnitude response is identical to that of $G(z)$ and has a minimum phase response.

$$G(z) = (2 - 3z^{-1})(3 + 4z^{-1})(2 + 5z^{-1})$$

$$\boxed{G(z) = 12(1 - \frac{3}{2}z^{-1})(1 + \frac{4}{3}z^{-1})(1 + \frac{5}{2}z^{-1})}$$

> 1



$-\frac{4}{3}$ $-\frac{3}{4}$

The minimum phase filter $P(z) = k(1-\frac{2}{3}z^{-1})(1+\frac{3}{4}z^{-1})(1+\frac{2}{5}z^{-1})$

Exercise:

The transfer function of a minimum-phase FIR filter is given by

$$G(z) = \left(-3+2z^{-1}\right)\left(4+3z^{-1}\right)\left(5+2z^{-1}\right)$$

From G(z) determine a transfer function of an FIR filter whose magnitude response is identical to that of G(z).

5.5 All-Pass Filters

An all-pass filter is one whose magnitude response is constant for all frequencies, but whose phase response is not identically zero.

[The simplest example of an all-pass filter is a pure delay system with system function $H(z) = z^{-k}$]

A more interesting all-pass filter is one that is described by

$$H(z) = \frac{a_L + a_{L-1}z^{-1} + \cdots + a_1 z^{-L+1} + a_0 z^{-L}}{1 + a_1 z^{-1} + \cdots + a_L z^{-L}},$$

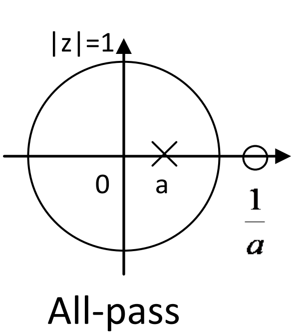
where $a_0 = 1$ and all coefficients are real. If we define the polynomial $A(z)$ as

$$A(z) = \sum_{k=0}^{L} a_k z^{-k} \qquad a_0 = 1$$

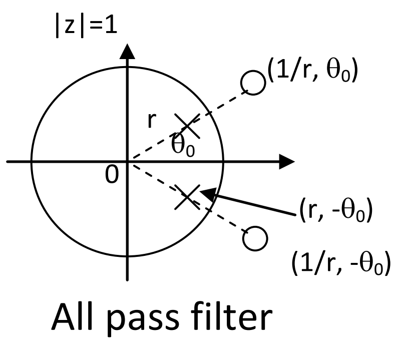
$$\therefore H(z) = z^{-L}\frac{A(z^{-1})}{A(z)} \Rightarrow |H(\theta)|^2 = H(z)\cdot H(z^{-1})|_{z=e^{j\theta}} = 1$$

i.e. all pass filter.

Furthermore, if $|z_0|$ is the modulus of a pole of $H(z)$, then $1/|z_0|$ is the modulus of a zero of $H(z)$ {i.e. the modulus of poles and zeros are reciprocals of one another}. The figure shown below illustrates typical pole-zero patterns for a single-pole, single-zero filter and a two-pole, two-zero filter.



All-pass



All pass filter

$$H(z) = \frac{1 - \frac{1}{a} z^{-1}}{1 - a z^{-1}} \qquad |a| < 1 \text{ for stability}$$

We can easily show that the magnitude response is constant.

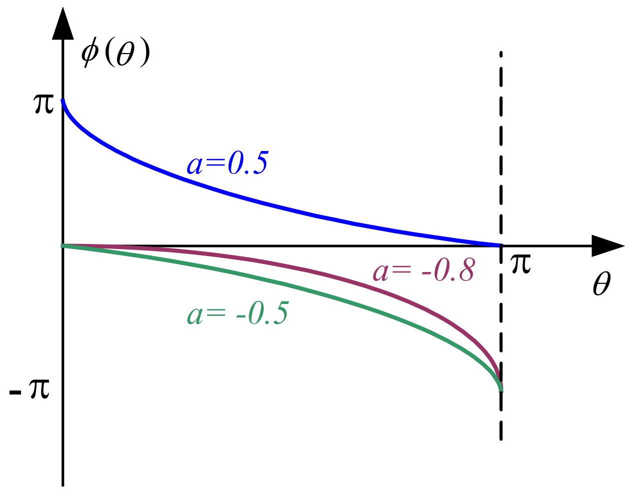
$$|H(\theta)|^2 = |H(\theta) \cdot H^*(\theta) = H(z) \cdot H(z^{-1})|_{z=e^{-j\theta}}$$

$$= \frac{1 - \frac{1}{a} e^{-j\theta}}{1 - a e^{-j\theta}} \cdot \frac{1 - \frac{1}{a} e^{j\theta}}{1 - a e^{j\theta}} = \frac{1 - \frac{2}{a}\cos\theta + \frac{1}{a^2}}{1 - 2a\cos\theta + a^2} = a^2$$

Phase response:

$$H(\theta) = \frac{1 - \frac{1}{a} e^{-j\theta}}{1 - a e^{-j\theta}} \cdot \frac{1 - a e^{j\theta}}{1 - a e^{j\theta}} = \frac{2 - (a + a^{-1})\cos\theta - j(a - a^{-1})\sin\theta}{1 - 2a\cos\theta + a^2}$$

$$\therefore \phi(\theta) = \tan^{-1}\left[\frac{-(a - a^{-1})\sin\theta}{2 - (a + a^{-1})\cos\theta}\right]$$



When $0 < a < 1$, the zero lies on the positive real axis. The phase over $0 \le \theta \le \pi$ is positive, at $\theta = 0$ it is equal to π and decreases until $\omega = \pi$, where it is zero.

When $-1 < a < 0$, the zero lies on the negative real axis. The phase over $0 \le \theta \le \pi$ is negative, starting at 0 for $\theta = 0$ and decreases to $-\pi$ at $\omega = \pi$.

Example: Show that the following transfer function $H_1(z)$ can be obtained using a parallel connection of two all-pass filters.

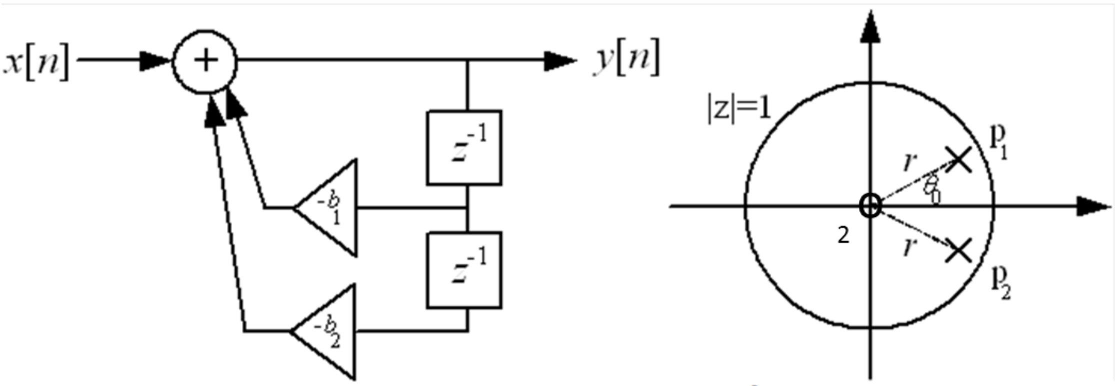
$$H_1(z) = \frac{10+6z^{-1}}{3+z^{-1}}$$

$$H_1(z) = \frac{9+3z^{-1}+1+3z^{-1}}{3+z^{-1}}$$

$$= 3 + \frac{1+3z^{-1}}{3+z^{-1}}$$

$$H_1(z) = \underbrace{3}_{All-pass\ filter} + \underbrace{\frac{1+3z^{-1}}{3+z^{-1}}}_{All-pass\ filter}$$

5.6 A second Order Resonant Filter



$$p_1 = re^{j\theta_0} = r\cos\theta_0 + jr\sin\theta_0$$
$$p_2 = re^{-j\theta_0} = r\cos\theta_0 - jr\sin\theta_0$$

$$H(z) = \frac{1}{1+b_1z^{-1}+b_2z^{-2}} = \frac{z^2}{z^2+b_1z+b_2} \qquad \text{(A)}$$

All pole systems has poles only (without counting the zeros as the origin)

$$H(z)=\frac{z^2}{z^2+b_1z^{-1}+b_2}=\frac{z^2}{(z-p_1)(z-p_2)}=\frac{z^2}{(z-re^{j\theta_0})(z-re^{-j\theta_0})}$$

$$H(z)=\frac{z^2}{z^2-r(e^{j\theta_0}+e^{-j\theta_0})z+r^2}=\frac{z^2}{z^2-2r\cos\theta_0 z+r^2} \qquad \text{(B)}$$

Comparing (A) and (B), we obtain

$b_1 = -2r\cos\theta_0$ and $b_2 = r^2$

$\therefore cos\theta_0 = -\dfrac{b_1}{2\sqrt{b_2}} \qquad \theta_0 = \dfrac{2\pi f_0}{f_s}$

θ_0= resonant frequency

Exercise:

An all poles system H(z) has two poles $p_{1,2}=0.8e^{\pm j\frac{\pi}{3}}$. Determine the transfer function of H(z) and sketch pole-zero plot of H(z).

5.7 Stability of a second-order filter

Consider a two-pole resonant filter given by

$$H(z)=\frac{z^2}{z^2+b_1z+b_2}=\frac{1}{1+b_1z^{-1}+b_2z^{-2}}$$

b_1 & b_2 are coefficients

This system has two zeros at the origin and poles at

$$p_1, p_2 = -\frac{b_1}{2}\pm\frac{\sqrt{{b_1}^2-4b_2}}{2}$$

- The filter is stable if the poles lies inside the unit circle i.e.
  $|p_1| < 1$ & $|p_2| < 1$

- For stability $b_2 < 1$. If $b_2 = 1$ then the system is an oscillator (Marginally stable)

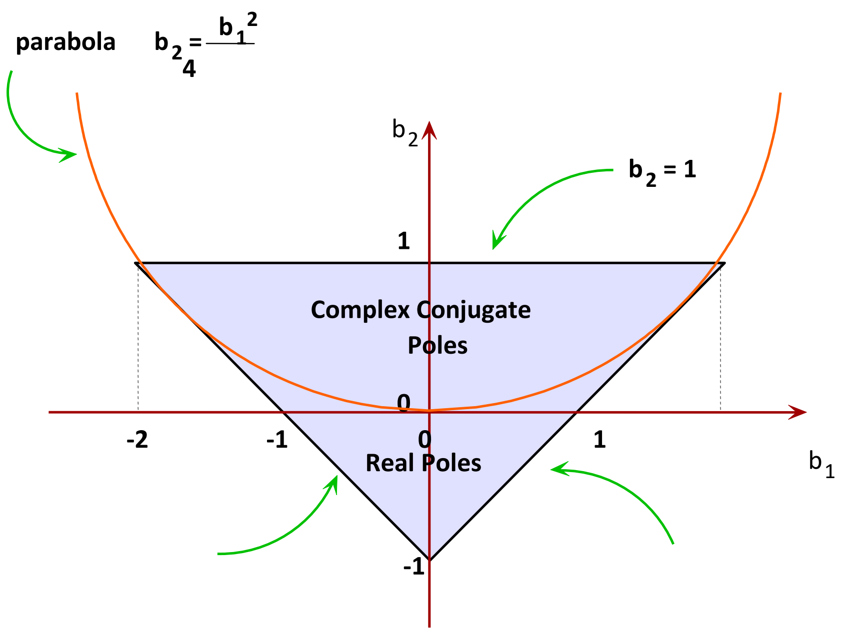
- Assume that the poles are complex

  ie. $b_1^2 - 4b_2 < 0 \Rightarrow b_1^2 < 4b_2$ and $b_1 < \pm 2\sqrt{b_2}$ $\quad b_2 > 0$

- If $b_1^2 - 4b_2 \geq 0$ then we get real roots.

The stability conditions define a region in the coefficient plane (b_1, b_2) which is in the form of a triangle (see below)

The system is only stable if and only if the point (b_1, b_2) lies inside the **stability triangle**.



Stability Triangle

If the two poles are real then they must have a value between -1 and 1 for the system to be stable.

$$-1 < \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} < 1$$

$$-2 + b_1 < \pm\sqrt{b_1^2 - 4b_2} < 2 + b_1$$

$$\therefore -2 + b_1 < -\sqrt{b_1^2 - 4b_2} \text{ and } \sqrt{b_1^2 - 4b_2} < 2 + b_1$$

$$(-2 + b_1)^2 > b_1^2 - 4b_2 \text{ and } b_1^2 - 4b_2 < (2 + b_1)^2$$

$$b_1 - b_2 - 1 < 0 \text{ and } b_1 + b_2 + 1 > 0$$

- The region below the parabola ($b_1^2 > 4b_2$) corresponds to real and distinct poles.
- The points on the parabola ($b_1^2 = 4b_2$) result in real and equal (double) poles.
- The points above the parabola correspond to complex-conjugate poles.

<u>Exerise:</u>

(a) Determine the stability region (triangle) for following the causal system

$$H(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

Your answer should include, sketch and equations of the boundaries of the region. Also, indicate the region where complex conjugate poles exist.

(b) The transfer function of a second order digital filter is given by $H(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}}$

Assuming $b_1^2 - 4b_2 < 0$ show that the resonant frequency (θ_0) of this filter is given by $\theta_0 = \cos^{-1}\left[\frac{-b_1}{2\sqrt{b_2}}\right]$

5.8 Digital Oscillators

A digital oscillator can be made using a second order discrete-time system, by using appropriate coefficients. A difference equation for an oscillating system is given by

$$p[n] = A\cos(n\theta)$$

From the table of z-transforms we know that the z-transform of $p[n]$ above is

$$P(z) = \frac{1 - \cos\theta z^{-1}}{1 - 2\cos\theta z^{-1} + z^{-2}}$$

Let $P(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 - \cos\theta z^{-1}}{1 - 2\cos\theta z^{-1} + z^{-2}}$,

Taking inverse z-transform on both sides, we obtain

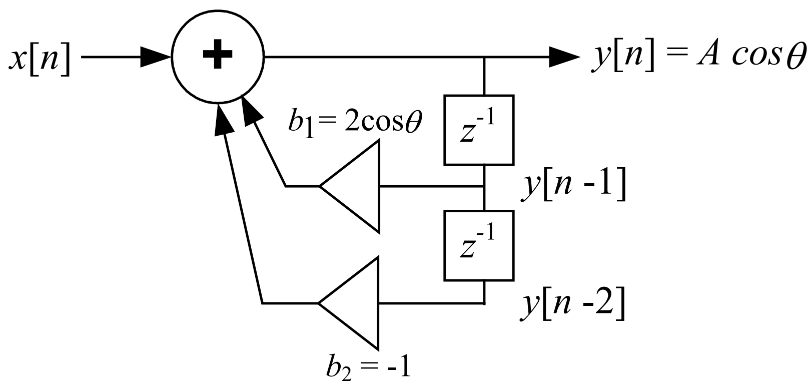
$$y[n] - 2\cos\theta y[n-1] + y[n-2] = x[n] - \cos\theta x[n-1]$$

An oscillator has no input. i.e., $x[n] = 0$ and $x[n-1] = 0$

So the equation of the digital oscillator becomes

$$y[n] = 2\cos\theta y[n-1] - y[n-2]$$

and its structure is shown below:



To obtain $y[n] = A\cos(n\theta)$, use the following initial conditions:

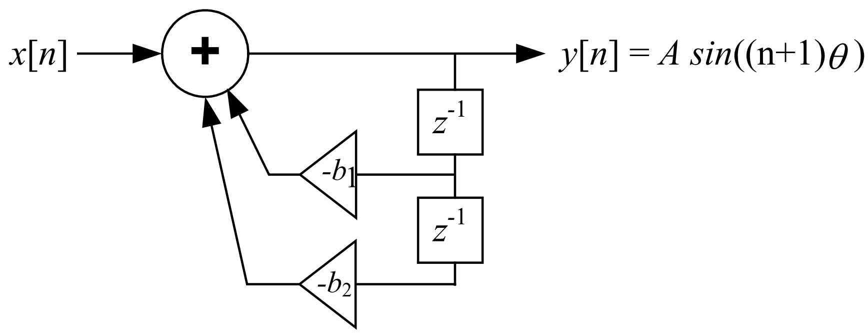
$$y[0] = A\cos(0.\theta) = A$$

$$y[-1] = A\cos(-1.\theta) = A\cos\theta$$

The frequency can be tuned by changing the coefficient b_1 (b_2 is a constant). The resonant frequency θ of the oscillator is,

$$\cos\theta = \frac{-b_1}{2\sqrt{b_2}} = -\frac{b_1}{2} \text{ (For an oscillator } b_2 = 1\text{)}$$

<u>Example:</u> A digital sinusoidal oscillator is shown below.



(a) Assuming θ_0 is the resonant frequency of the digital oscillator, find the values of b_1 and b_2 for sustaining the oscillation.

$$H(z) = \frac{K}{1 + b_1 z^{-1} + b_2 z^{-2}} = \frac{K}{(z - re^{j\theta_0})(z - re^{-j\theta_0})} = \frac{K}{z^2 - r(e^{j\theta_0} + e^{-j\theta_0})z + r^2}$$

$$= \frac{K}{z^2 - 2r\cos\theta_0 z + r^2}$$

$$\therefore b_1 = -2\, r\cos\theta_0; \qquad b_2 = r^2$$

For oscillation $\quad b_2 = 1 \Rightarrow r = 1 \qquad \therefore\ b_1 = -2\cos\theta_0$

(b) Write the difference equation for the above figure. Assuming

$$x[n] = (A\sin\theta_0)\delta[n], \text{ and } y(-1) = y(-2) = 0.$$

Show, by analysing the difference equation, that the application of an impulse at $n = 0$ serves the purpose of beginning the sinusoidal oscillation, and prove that the oscillation is self-sustaining thereafter.

$$y[n] = -b_1 y[n-1] - y[n-2] + x[n]$$
$$y[n] = 2\cos\theta_0 y[n-1] - y[n-2] + A\sin\theta_0\delta[n]$$

$n = 0$

$$y[0] = 2\cos\theta_0\, \underset{0}{\underset{\uparrow}{y[-1]}} - \underset{0}{\underset{\uparrow}{y[-2]}} + A\sin\theta_0\, \underset{0}{\underset{\uparrow}{\delta[0]}}$$

$$y[0] = A\sin\theta_0$$

$n = 1$

$$y[1] = 2\cos\theta_0\, y[0] - \overset{0}{y[-1]} + A\sin\theta_0\, \overset{0}{\delta[1]}$$

$$y[1] = 2\cos\theta_0\, A\sin\theta_0 = A\sin 2\theta_0$$

$n = 2$

$$\begin{aligned} y[2] &= 2\cos\theta_0\, y[1] - y[0] + A\sin\theta_0\delta[2] \\ &= 2\cos\theta_0\, A\sin 2\theta_0 - A\sin\theta_0 \\ &= 2A\cos\theta_0\,[2\sin\theta_0\cos\theta_0] - A\sin\theta_0 \\ &= A\sin\theta_0\,[4\cos^2\theta_0 - 1] = A[3\sin\theta_0 - 4\sin^3\theta_0] \end{aligned}$$

where $\sin 3\theta_0 = 3\sin\theta_0 - 4\sin^3\theta_0$

$y[2] = A\sin 3\theta_0$ and so forth

(c) By setting the input to zero and under certain initial conditions, sinusoidal oscillation can be obtained using the structure shown above. Find these initial conditions.

$$y[n] = 2\cos\theta y[n-1] - y[n-2] + x[n]$$

($x[n]$ = 0 for an oscillator)

$n = 0$ $\quad$ $y[0] = 2\cos\theta_0\, y[-1] - y[-2]$

for oscillation, $y[-1] = 0 \Rightarrow$ no cosine terms

$y[0] = -y[-2] \Rightarrow y[-2] = -A\sin\theta_0$ (sine term is required)
$y[0] = 0-(-A\sin\theta_0) = A\sin\theta_0$

Initial conditions: $y[-1] = 0; y[-2] = -Asin\theta_o$

Exercise:

The transfer function of a second order digital oscillator with an oscillation frequency of θ_0 is given by

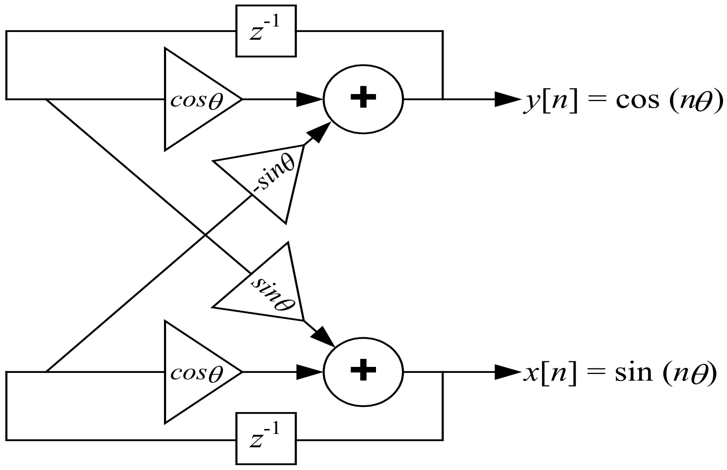
$$H(z) = \frac{z}{z^2 - b_1 z + 1}; b_1 = 2\cos\theta_0$$

Show that the impulse response corresponding to the system function H(z) is given by

$$h(n) = \frac{\sin(n\theta_0) \quad u(n)}{\sin(\theta_0)}$$

5.8.1 Sine and cosine oscillators

Sinusoidal oscillators can be used to deliver the carrier in modulators. In modulation schemes, both sines and cosines oscillators are needed. A structure that delivers sines and cosines simultaneously is shown below:



A simultaneous sine and cosine oscillator

<u>Proof:</u>

Trigonometric equation for $cos(n+1)\theta$ is:

$$\cos((n+1)\theta) = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)$$

Let $y[n] = \cos(n\theta)$ and $x[n] = \sin(n\theta)$

$$\therefore y[n+1] = \cos(\theta)\, y[n] - \sin(\theta)\, x[n]$$

Replace n by n-1: $y[n] = \cos(\theta)\, y[n-1] - \sin(\theta)\, x[n-1]$ (A)

Similarly

$$\sin((n+1)\theta) = \sin(\theta)\cos(n\theta) + \sin(n\theta)\cos(\theta)$$

$$\therefore x[n+1] = \sin(\theta)\, y[n] + x[n]\cos(\theta)$$

Replace $n \rightarrow n$-1

$$x[n] = \sin(\theta)\, y[n-1] + x[n-1]\cos(\theta) \qquad \text{(B)}$$

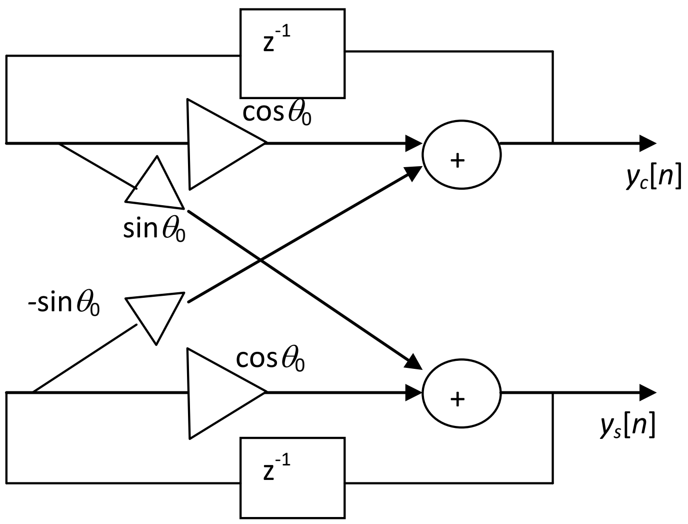
Using equations A & B above, the structure shown above can be obtained.

Example: An oscillator is given by the following coupled difference equations expressed in matrix form.

$$\begin{bmatrix} y_c[n] \\ y_s[n] \end{bmatrix} = \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} y_c[n-1] \\ y_s[n-1] \end{bmatrix}$$

Draw the structure for the realisation of this oscillator, where θ_0 is the oscillation frequency. If the initial conditions $y_c[-1] = A\cos\theta_0$ and $y_s[-1] = -A\sin\theta_0$, obtain the outputs $y_c[n]$ and $y_s[n]$ using the above difference equations.

$$y_c[n] = \cos\theta_0 y_c[n-1] - \sin\theta_0 y_s[n-1]$$
$$y_s[n] = \sin\theta_0 y_c[n-1] + \cos\theta_0 y_s[n-1]$$



n=0 $\quad y_s[0] = \sin\theta_0 (A \cos\theta_0) + \cos\theta_0 (-A\sin\theta_0) = 0$

n=0 $\quad y_c[0] = \cos\theta_0 (A\cos\theta_0) - \sin\theta_0 (-A\sin\theta_0) = \boldsymbol{A}$

n=1 $\quad y_c[1] = \cos\theta_0 . A - \sin\theta_0 .0 = \boldsymbol{A\cos\theta_0}$

n=1 $\quad y_s[1] = A \sin\theta_0 + 0 = A \sin\theta_0$

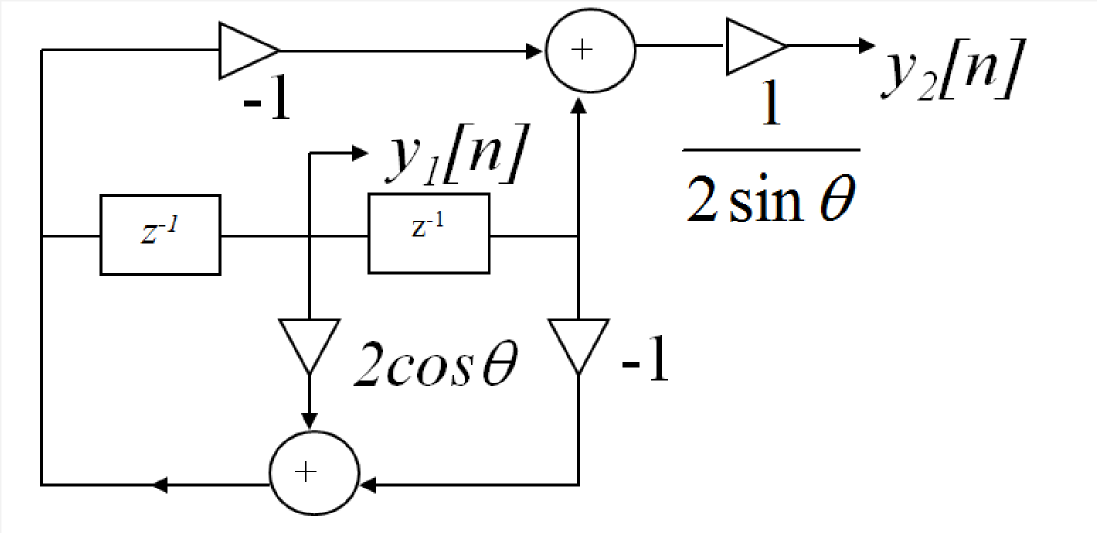
n=2 $\quad y_c[2] = \cos\theta_0\, y_c[1] - \sin\theta_0\, y_s[1]$

$\quad = \cos\theta_0\, A \cos\theta_0 - \sin\theta_0\, A \sin\theta_0 = \boldsymbol{A \cos(2\theta_0)}$

$n = n \quad y_c[n] = \boldsymbol{A \cos (n\theta_0)}$

similarly $\quad y_s[n] = A \sin(n\theta_0)$

Exercise: For the structure shown below, write down the appropriate difference equations and hence state the function of this structure.



CHAPTER 5: PROBLEM SHEET 5

Q1)

a) Show that both digital filters given below have the same magnitude response:

i) $y[n] = \sum_{i=-m}^{m} c_i x[n-i]$ ii) $y[n] = \sum_{i=-m}^{m} c_i x[n-m-i]$ i)

$y[n]$ = output; $x[n]$ = input; c_i = coefficients

b) Compute the 3dB bandwidth of the following filters

$$H_1(z) = \frac{1-a}{1-az^{-1}}; H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}}, 0 < a < 1$$

Which filter has a smaller 3dB bandwidth?

Ans: $\theta_{c_1} = \cos^{-1}\left[\frac{4a-a^2-1}{2a}\right]$; $\theta_{c_2} = \cos^{-1}\left[\frac{2a}{1+a^2}\right]$; $\therefore \theta_{c_2} < \theta_{c_1} \Rightarrow 2^{nd}$ *filter.*

c) Find the magnitude response for the system function *H(z)* and comment on your result

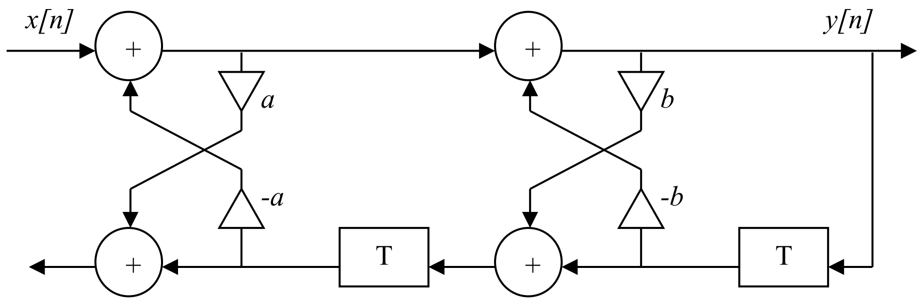
$$H(z) = \frac{1+3z^2}{3+z^2}$$

Draw the canonic realization of the system *H(z)*.

Ans: $|H(\theta)| = 1$ allpass filter

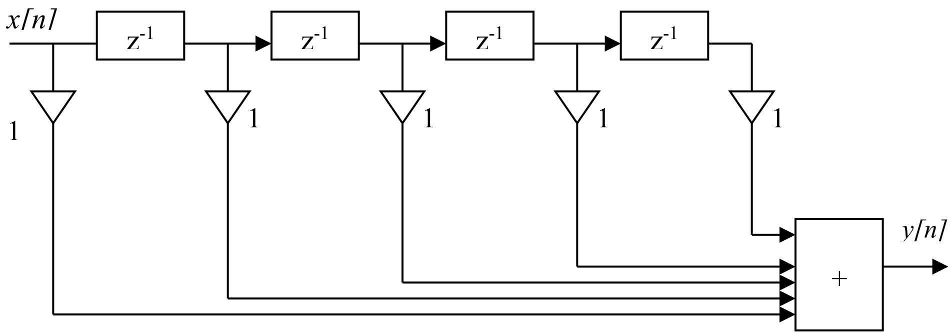
Q2)

a) Determine the frequency response $H(\theta)$ of the lattice filter



$$Ans: H(\theta) = \frac{1}{1+b(a+1)e^{-j\theta} + ae^{-j2\theta}}$$

b) Show that the filter structure shown below has a linear phase characteristic equation given by: $\phi(\theta) = -2\theta$.



Q3)

Consider the following causal IIR transfer function:

$$H(z) = \frac{2z^3 - 4z^2 + 9}{(z-3)(z^2 + z + 0.5)}$$

Is $H(z)$ a stable function? If it is not stable, find a stable transfer function $G(z)$ such that $|G(\theta)| = |H(\theta)|$. Is there any transfer function having the same magnitude response as $H(z)$?

Ans: unstable,

$$G(z) = \frac{2z^3 - 4z^2 + 9}{(1-3z)(z^2 + z + 0.5)} = H(z)A(z)$$

$$|G(z)| = |H(\theta)||A(\theta)|, A(z) = \frac{1-3z}{z-3} \quad \{Allpass\ filter\}$$

Q4)

a) A two-pole lowpass filter has the system function

$$H(z) = \frac{k_0}{(1 - b_1 z^{-1})^2}$$

Determine the values of k_0 and b_1 such that the frequency response $H(\theta)$ satisfy the conditions

$$H(0) = 1; \quad \left|H(\theta)\right|^2_{\theta = \frac{\pi}{4}} = \frac{1}{2}$$

Ans: k=0.46, b_1 = 0.32

b) A third order FIR filter has a transfer function $G(z)$ given by

$$G(z) = 30(1 - \frac{3}{2}z^{-1})(\frac{4}{3} + z^{-1})(1 + \frac{5}{2}z^{-1})$$

From $G(z)$, determine the transfer function of an FIR filter whose magnitude response is identical to that of $G(z)$ and has a maximum phase response.

Q5)

Obtain a parallel realization for the following $H(z)$

$$H(z) = \frac{z^2 - 4z + 3}{z(z+0.5)^2}$$

Implement the parallel realization of *H(z)* which you have obtained

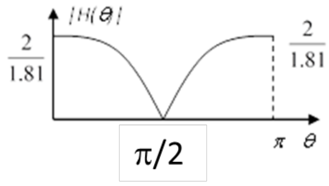
Ans: $$H(z) = 12z^{-1} + \frac{-11z^{-1} - 16z^{-2}}{1 + z^{-1} + \frac{1}{4}z^{-2}}$$

Q6) A notch filter is given by

$$H(z) = \frac{1 + z^{-2}}{1 + 0.8z^{-2}}$$

Determine the frequency response at DC, $\frac{f_s}{4}$ and $\frac{f_s}{2}$ (f_s – sampling frequency). Sketch the frequency response in the interval $0 \le f \le \frac{f_s}{2}$

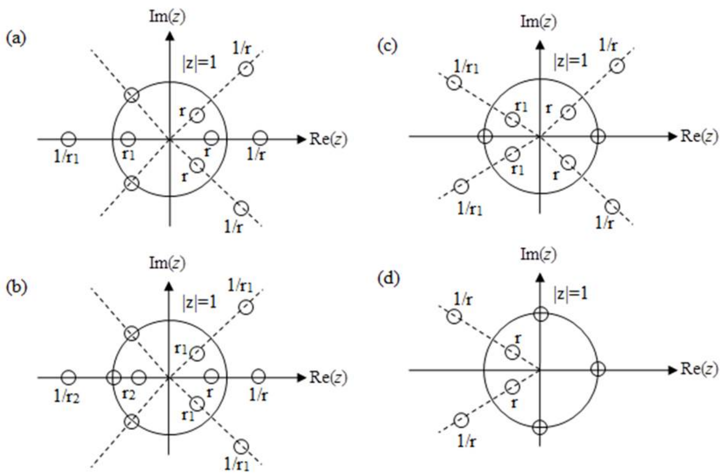
Ans:



Q7) Find all of the zeros of type 2 linear-phase sequence, if it is known that there are zeros at $z = \frac{1}{4}e^{j\frac{\pi}{6}}$ and $z = 1$.

Ans: $z = \frac{1}{4}e^{j\frac{\pi}{6}}$, $z = \frac{1}{4}e^{-j\frac{\pi}{6}}$, $z = 4e^{j\frac{\pi}{6}}$, $z = 4e^{-j\frac{\pi}{6}}$

Q8) Typical plots of zeros for linear-phase systems are shown below. From the pole-zero plot, identify the sequence type



Q9) Determine and sketch the magnitude and phase response of the following filters:

(i) $y[n] = \frac{1}{2}(x[n] - x[n-1])$

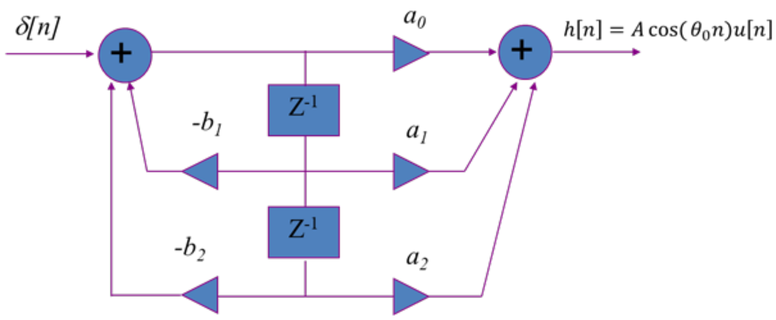
(ii) $y[n] = x[n] - x[n-8]$

(iii) $y[n] = x[n-4]$

Q10) The block diagram shows a digital oscillator which is initialized by an impulse as shown in the figure below. The desired unit impulse response is:

$$h[n] = A\cos(\theta_0 n)u[n]$$

Assuming θ_0 is the resonant frequency of the digital oscillator and writing appropriate equations find the values of a_0, a_1, a_2, b_1 and b2.



End of Chapter 5